

# On Spatial Point Processes with Uniform Births and Deaths by Random Connection

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## Abstract

This paper is focused on a class of spatial birth and death process of the Euclidean space where the birth rate is constant and the death rate of a given point is the shot noise created at its location by the other points of the current configuration for some response function  $f$ . An equivalent view point is that each pair of points of the configuration establishes a random connection at an exponential time determined by  $f$ , which results in the death of one of the two points. We concentrate on space-motion invariant processes of this type. Under some natural conditions on  $f$ , we construct the unique time-stationary regime of this class of point processes by a coupling argument. We then use the birth and death structure to establish a hierarchy of balance integral relations between the factorial moment measures. Finally, we show that the time-stationary point process exhibits a certain kind of repulsion between its points that we call  $f$ -repulsion.

## 1 Introduction

This paper introduces a class of motion invariant <sup>1</sup> on the Euclidean space where births take place according to some homogeneous Poisson rain and where the instantaneous death rate of a point of the current configuration is the shot-noise [2] of the configuration at this point for some positive response function  $f$ .

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<sup>1</sup>By motion invariant, we mean spatial birth and death processes invariant by all translations and rotations of the Euclidean space.

The analysis of these dynamics was initially motivated by models arising in peer to peer networking [3]. This class of processes is however of more general potential interest as it features non-trivial interactions between points combining “density” and “geometry” components that should have other practical incarnations. Here, this comes from the fact that the death rate cannot be assessed through densities only as connections are also functions of distances. In fact, the presence of a point at some location implies that there are less points around than what a density argument would predict. This Palm type bias [4] makes the role of geometry quite central.

The main mathematical results on this class of processes are (i) an existence and uniqueness result on their stationary regimes (Theorem 7); (ii) a hierarchy of balance equations linking their factorial moment measures [4] of neighboring orders (Theorem 8); (iii) a general repulsion result formalizing the Palm bias alluded to above (Theorem 9).

These results can be seen as a complement to those of Garcia and Kurtz in [5]. In this last paper, the authors also considered a spatial birth and death process on the whole Euclidean space but in the case where the birth rate (rather than the death rate here) depends on the configuration.

The first sections of the paper are focused on the construction of the stationary regime of such spatial birth and death processes. There is no fundamental difficulty in building such a stationary regime when the phase space is compact using the formalism of Preston [1] together with Markovian techniques. The main challenge addressed here is hence that of the construction when the phase space is the whole Euclidean space. A pathwise construction of this steady state is proposed. The first step of this approach consists in building the state for all compacts of time and a large enough class of initial conditions. This construction leverages the random connection interpretation of the shot noise death process. It consists in a recursive investigation for determining which connection is responsible of each individual death. It is defined in Section 3 and is called *Sheriff*. The second step builds a coupling between the dynamics with an empty initial condition and that with a motion invariant initial condition  $\mathcal{Z}_0$ . This construction, which is defined in Section 4 is again pathwise and a natural extension of Sheriff which is called Sheriff<sup>Z</sup>. We then show that, in the coupling of Sheriff<sup>Z</sup>, the influence of each point of  $\mathcal{Z}_0$  almost surely dies out in finite time. This is based on martingale and random walk arguments which are given at the end of Section 4. Tightness and positiveness arguments (Section 5) are then combined with differential equations on the densities and on second moment measures (Section 6) to prove that on any compact of space, the time of last influence of  $\mathcal{Z}_0$  is actually integrable. This allows

one to develop a coupling from the past argument (Section 7) which proves the existence and uniqueness result.

Section 8 gathers a few basic properties on the stationary distribution in question. We show that the differential equations alluded to above lead to a set of conservation laws for moment measures that mimic the Markov birth and death structure: the  $k$ -th moment measure is balanced by certain integral forms of the  $k - 1$ -st and the  $k + 1$ -st, with the usual reflection at  $k = 0$ . The stationary regime is also shown to exhibit  $f$ -repulsion, a property which translates the fact that a typical point of the stationary configuration suffers of a smaller death rate than that seen by the typical locus of the Euclidean space.

Lastly, In the appendix, (Section 9), we detail the proofs of some of the Equations and provide a table of notation.

The model and its dual representation in term of either shot-noise or random connections is described in the following Section.

## 2 Model

We start with two informal definitions of the stochastic process of interest. We then give a formal definition of the problem.

### 2.1 Model Description

#### 2.1.1 Spatial Birth and Death Viewpoint

Let  $D$  be a closed convex set of  $\mathbb{R}^d$ . Let  $M(D)$  denote the set of counting measures  $\phi$  on  $D$  (see e.g. [6]). Depending on the situation, we will consider the point process  $\phi$  either as a counting measure, or as a set, the support of this counting measure. As a result, the number of points of  $\phi$  in the Borel set  $C$  will be denoted either  $\phi(C)$  or  $|\phi \cap C|$ , with  $|S|$  the cardinality of the set  $S$ , depending on the circumstances. Let  $\mathcal{M}(D)$  denote the smallest  $\sigma$ -field containing all the events  $\phi(C) = k$ ,  $C$  ranging over Borel subsets of  $D$  and  $k$  over integers.

We consider a *spatial birth and death* (SBD) process on  $D$ , namely a  $M(D)$ -valued Markov jump process [1]. The state (or point configuration) of this Markov process at time  $t$  will be denoted by  $\Phi_t \in M(D)$ .

It is well known [1] that when  $D$  is compact, such a Markov process is characterized by two rate functionals, the birth rate functional  $b(\phi, \phi + \delta_x)$ , which gives the infinitesimal rate of a birth at  $x \in D$  in configuration  $\phi \in$

$M(D)$  and the death rate functional  $\mu(\phi + \delta_x, \phi)$ , which gives the infinitesimal rate of the death of  $x \in \phi + \delta_x$  in configuration  $\phi + \delta_x \in M(D)$ .

The birth rate functional considered in the present paper is homogeneous in time and space, namely

$$b(\phi, \phi + \delta_x) = \lambda, \quad (1)$$

for all  $x \in D$  and  $\phi \in M(D)$ , where  $\lambda$  is a positive real number.

Let  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  be a non-negative function which we will call the *response function* of the model. The death rate of the SBD process considered in the present paper is determined by this function through the relation

$$\mu(\phi + \delta_x, \phi) = \sum_{y \in \phi} f(\|x - y\|). \quad (2)$$

It is homogeneous in time but not in space: the death rate of  $x$  in configuration  $\phi + \delta_x$  is the *shot noise* created by  $\phi$  at  $x$  for the response function  $f$ .

In the compact domain case the finite time horizon problem can be analyzed by classical Markov chain uniformization techniques and the existence and the uniqueness of the time stationary regimes can be proved using the theory of Markov chains in general state spaces [7].

The object of interest in this paper is the extension of these dynamics to  $\mathbb{R}^d$ . When  $D = \mathbb{R}^d$ , the above Markov approach fails even for the construction of the finite time horizon state.

### 2.1.2 Death by Random Connection Viewpoint

Another equivalent description of the dynamics is in terms of a Random Connection Graph (RCG). A RCG [8, 9] on a point process  $\Phi \in M(D)$  is informally defined as follows: for all  $\Phi \in M(D)$ , for all unordered pairs  $\{x, y\}$  of points of  $\Phi$ , one samples an independent Bernoulli random variable  $Q(x, y)$  with value 1 with probability  $c(\|x - y\|)$  and 0 with probability  $1 - c(\|x - y\|)$ . The function  $c : \mathbb{R}^+ \rightarrow [0, 1)$  will be referred to as the connection function. The associated random connection model is the random graph on  $\phi$  with edges between the points  $(x, y)$  such that  $Q(x, y) = 1$ .

Informally, the SBD process studied in this paper can also be obtained by sampling, for all unordered pairs  $\{x, y\}$  of points of  $\Phi$ , an independent exponential random variable  $T_{xy}$  with rate  $2f(\|x - y\|)$  and in establishing at this time a lethal connection between  $x$  and  $y$  which instantly kills either of the two with probability  $1/2$ , independently of everything else. This death

can, however, only happen if the points  $x$  and  $y$  are still alive at time  $T_{xy}$ , which is not guaranteed as each might have already been killed by other points. It should be clear that, at least in the case where  $D$  is compact and the time interval is compact as well, the death rate of any given point  $x \in \phi$  is then given by (2) as the deaths that occur in state  $\phi$  in an infinitesimal interval of time with length  $dt$  can be obtained by sampling with probability  $1/2$  the points connected by edges in a RCG on  $\phi$  with connection function  $c(r) = 2f(r)dt$ . In view of this, it makes sense to call this mechanism *death by random connection*.

This second view point will be instrumental for constructing the process on  $\mathbb{R}^d$ .

## 2.2 Problem Statement

We will represent the births as a  $\mathbb{R}^d \times \mathbb{R}$  Poisson point process,  $\Psi$ , i.e. the births in the time interval  $(t_0, t_1)$  are  $\Psi_{(t_0, t_1)} = \Psi_{\mathbb{R}^d \times (t_0, t_1)}$ , a Poisson process on  $\mathbb{R}^d \times (t_0, t_1)$  with intensity measure  $\lambda l^d \times l(t_0, t_1)$ , where  $l^d$  (resp.  $l$ ) stands for the Lebesgue measure of  $\mathbb{R}^d$  (resp.  $\mathbb{R}$ ). A point  $p \in \Psi$  will also be denoted by  $(x_p, b_p)$ , with  $x_p \in \mathbb{R}^d$  the location of the birth and  $b_p \in (t_0, t_1)$  the time of the birth. The point process  $\Psi_{\mathbb{R}^d \times (t_0, \infty)}$  will be denoted by  $\Psi_{t_0}$ .

For any two points  $p, q \in \Psi$ , let  $I_{pq}$  and  $T_{pq}$  be two random variables independent of everything else with distributions

$$\begin{aligned} I_{pq} &= 1 - I_{qp} \sim \text{Bernoulli}\left(\frac{1}{2}\right), \\ T_{pq} &= T_{qp} \sim (b_p \vee b_q) + \text{Exp}(2f(\|x_p - x_q\|)). \end{aligned}$$

These quantities have the interpretations alluded to above:

- $T_{pq}$  is the time at which the connection between  $p$  and  $q$  is realized (it will actually be the death of one of them, if both are alive just before  $T_{pq}$ );
- $I_{pq} = 1$  if the direction of the connection is from  $q$  to  $p$  (the dying point will be  $p$  if both are alive just before  $T_{pq}$ ;  $q$  is then said to kill  $p$  at time  $T_{pq}$ ).

As long as both points are alive, they both “feel” a (time) intensity  $f(\|x_p - x_q\|)$  to be killed by the other.

The dynamics of interest can be defined by the equation:

$$d_p = \inf \{T_{pq} : q \in \Psi, d_q \geq T_{pq}, I_{pq} = 1\}. \quad (3)$$

Note that the condition  $d_q \geq T_{pq}$  makes the definition recursive with respect to time. Here is a continuous time version of the last equation in terms of a stochastic differential equation: for all bounded sets  $C$  of  $\mathbb{R}^d$ ,

$$d\Phi_t(C) = \Psi(C, dt) - \sum_{X \in \Phi_t} \sum_{Y \neq X \in \Phi_t} \delta_X(C) N(X, Y, dt), \quad (4)$$

where  $t \rightarrow \Psi(C, t)$  is a Poisson point process of intensity  $\lambda|C|$  on  $\mathbb{R}$  and  $t \rightarrow N(x, y, t)$ ,  $x, y \in \mathbb{R}^d$ , is a collection of independent Poisson point processes with  $N(x, y, t)$  of intensity  $f(\|x - y\|)$  on  $\mathbb{R}$  for all  $x, y \in \mathbb{R}^d$ . Note that in each realization only a countable number of these Poisson processes comes to the scene.

The general problem can be stated in the following terms: given some initial condition  $\Phi_0$ , which is some point process in  $\mathbb{R}^d$ , (i) can one construct a solution  $\{\Phi_t\}$  to (4) where  $\Phi_t$  is a point process on  $\mathbb{R}^d$  for all  $t > 0$ ? (ii) if so, under what conditions does  $\Phi_t$  converge in distribution to a limit? (iii) does this limit, when it exists, depend on the initial condition?

### 2.3 Assumptions on the Response Function

Throughout the paper, when  $D = \mathbb{R}^d$ , the following properties on  $f$  will be considered:

- **Assumption 0:**  $f$  is non negative and  $f(0) = 0$ .
- **Assumption 1**

$$0 < a < \infty, \text{ where } a := \int_{\mathbb{R}^d} f(\|x\|) dx. \quad (5)$$

- **Assumption 2:** the function  $r \rightarrow f(r)$  is monotone non-increasing on  $(0, \infty)$ .
- **Assumption 3:** the function  $f$  is bounded above. We will then denote by  $K$  the upper-bound on  $f$ .

Assumption 0 is natural in this context; the assumption that  $f(0) = 0$  makes sense as we always deal with simple point processes. Assumption 1 is used throughout the paper. This assumption is used for proving that events can be sorted out in  $\mathbb{R}^d$  (Lemma 1 and Theorem 1). Assumption 2–3 are only needed in the final steps of the construction of the stationary regime; they are not required for the construction on compacts of time. If Assumption 2 holds (which we do not assume in general), the death rate is higher in regions with many points.

### 3 Construction on Finite Time Horizon

The main question addressed in this section is whether there exists a solution to (4) (or equivalently to (3)). We use the connection–death view point described in Sections 2.1.2 and 2.2 to construct these dynamics pathwise over all compacts of time and space.

It should first be noticed that (3) may be problematic if the set

$$N_p = \{T_{pq} : q \in \Psi_{t_0}\}$$

has accumulation points. The following proposition gives a condition guaranteeing that this is a.s. not the case when  $t_0$  is finite.

**Lemma 1.** *Assume that  $t_0 > -\infty$  and that Assumptions 0–1 hold. Then almost surely none of the sets  $N_p$ ,  $p \in \Psi_{t_0}$ , has accumulation points.*

*Proof.* For any  $p = (x, t) \in \mathbb{R}^d \times [t_0, \infty)$ , the conditional distribution of  $\Psi_{t_0} - \delta_p$  given that  $\Psi_{t_0}$  has a point in  $p$  is a Poisson process with same distribution as  $\Psi_{t_0}$  (Slivnyak’s theorem). In the following,  $\mathbb{E}^p$  denotes this conditional expectation, or equivalently the Palm distribution of  $\Psi_{t_0}$  at  $p$ .

To each point  $q = (y, s)$  of  $\Psi_{t_0}$ , we associate the point  $T_{pq}$  of  $\mathbb{R}$ .

We show that the intensity measure of this point process on  $\mathbb{R}$  is locally finite under the condition given above. For any  $u \geq t$ ,

$$\begin{aligned} \mathbb{E}^p[N_p \cap (t_0, u]] &= \mathbb{E}^p \int_{\mathbb{R}^d \times (t_0, \infty)} 1_{\{T_{pq} \leq u\}} (\Psi_{t_0} - \delta_p)(dq) \\ &= \mathbb{E} \int_{\mathbb{R}^d \times (t_0, u]} 1_{\{T_{pq} \leq u\}} (\Psi_{t_0})(dq) \\ &= \lambda \int_{\mathbb{R}^d} \int_{t_0}^u \mathbb{P}(\text{Exp}(f(\|x - y\|)) \leq u - (t \vee v)) dv dy \\ &\leq \lambda(u - t_0) \int_{\mathbb{R}^d} \left(1 - e^{-(u-t_0)f(\|y\|)}\right) dy \\ &\leq \lambda(u - t_0)^2 \int_{\mathbb{R}^d} f(\|y\|) dy = \lambda(u - t_0)^2 a < \infty. \end{aligned}$$

Here  $\text{Exp}(z)$  denotes an exponential random variable of parameter  $z$ ; the third equality is Campbell’s formula; we used the inequality  $1 - e^{-z} \leq z$  in the last line.  $\square$

**Remark 1.** *The last lemma holds under the weaker assumption  $a(1) := \int_{\mathbb{R}^d \setminus B(0,1)} f(\|x\|) dx < \infty$ .*

Thus, for all  $p$ , every finite interval of  $[t, \infty)$  contains an a.s. finite number of points of the type  $T_{pq}$ ,  $q \in \Psi_{t_0}$ . Note that Lemma 1 fails with  $t_0 = -\infty$ .

### 3.1 The Sheriff Algorithm

To construct the death process when  $t_0 > -\infty$  and  $t_1 < \infty$ , we propose below an algorithm that we call the Sheriff algorithm. Its name comes from the following Western imagery: there is wild shooting in an infinite saloon of  $\mathbb{R}^d$  with cowboys arriving over time and space; the sheriff has to find out who is still alive at a given time and who was killed by whom before this time.

Within the setting of Section 2.2, the general idea is quite natural: one picks a node, checks its earliest connection (potential death) time; in order to determine whether this is its actual death time, one has to determine whether the death time of the killer is earlier or later than this time (Equation (3)); for this, one checks the earliest connection time of the latter, etc.

#### Algorithm 1.

The Sheriff Algorithm: Construction of the death process on time interval  $(t_0, t_1)$ .

1 Initialization:

- Every point born in  $(t_0, t_1)$ , say  $p = (x, t)$  with  $t_0 \leq t \leq t_1$  has a stack of its *death sentences*. A death sentence for  $p$  is a triple  $(p, q, T_{pq})$ , where  $q$  is a potential killer of  $p$ , i.e.  $I_{pq} = 1$ . Death sentences are sorted earliest on top<sup>2</sup>;
- The sheriff has an investigation stack, initially empty.

2 If the investigation stack is empty, the sheriff chooses, from a pre-defined ordering of all points<sup>3</sup>, the first point whose stack has on top a death sentence with time less than  $t_1$  and no death certificate, and moves the sentence to the investigation stack. If there is no such point, then the procedure ends<sup>4</sup>.

3 The sheriff looks at the sentence on top of the investigation stack, say  $(p, q, T)$ , and does one of the following:

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<sup>2</sup>Note that, under the assumptions of Lemma 1, each stack has simple sequential order.

<sup>3</sup>By this, we mean a bijection between the points of the configuration and  $\mathbb{N}$ ; for instance, points can be sorted in function of their distance to the origin of the Euclidean space, and ties, if any, can be solved in a random way.

<sup>4</sup>We shall see that, under our assumptions of an infinite domain, the sheriff never stops.



- If killer  $q$ 's stack has on top a death sentence or death certificate with a time larger than  $T$ , then  $q$  is alive at  $T$  and the execution happens. The sheriff changes the sentence  $(p, q, T)$  into a death certificate with the same data and returns it to the top of  $p$ 's stack.
- If  $q$  has a death certificate earlier than  $T$ , then the execution is not realized and the sentence  $(p, q, T)$  is discarded, i.e. the investigation stack is popped.
- Otherwise the sheriff moves the top sentence of  $q$ 's stack to the investigation stack.

4 Go to 2.

To prove that the Sheriff Algorithm works properly, we start with the following lemma which shows that for all finite intervals  $(t_0, u)$  as above, for all predefined ordering of the points, for all cards, the recursive investigation performed by Sheriff to determine the status of this card ends in finite time.

**Lemma 2.** *Assume that  $t_0 > -\infty$  and that  $f$  satisfies Assumptions 0–1. Then, almost surely, there is no infinite sequence of points  $p_1, p_2, \dots$  such that the sequence  $T_{p_n p_{n+1}}$  is non-increasing.*

*Proof.* Notice that since  $t_0 > -\infty$ , the sequence  $T_{p_n p_{n+1}}$  is bounded from below.

The proof uses percolation properties of the Poisson RCG in  $\mathbb{R}^d$  [8, 9]. Let us view the arrival locations  $x_p$  of the points  $p$  of  $\Psi_{t_0}$  arrived until time  $u$  as a homogeneous Poisson  $\Phi$  point process of  $\mathbb{R}^d$  with intensity  $\lambda(u - t_0)$ . The time of arrival  $b_p$  of point  $p$  is seen as an independent mark, uniform on  $(t_0, u)$ .

Let  $J$  be some time interval of  $(t_0, u)$ . We create an undirected edge (a connection) between the points  $x_p$  and  $x_q$  of  $\Phi$  if  $T_{pq} \in J$ . This does not form a RCG because of the marks (in the RCG, one establishes an edge between two points of a Poisson point process with a probability that depends on their distance only; here the mark of point  $x_p$  creates a correlation between the edges that connect  $x_p$  to the other points).

Consider now the model where one creates an undirected edge between  $x_p$  and  $x_q$  of  $\Phi$  if  $M_{x_p x_q}(J) > 0$  where  $M_{x_p x_q}$  is a Poisson point process on  $\mathbb{R}$  with intensity  $f(\|x_p - x_q\|)$ , conditionally independent of everything else given  $\|x_p - x_q\|$ . By a standard coupling argument, this defines a *dominating RCG*, i.e. a RCG where there are more edges than in the original model.

The mean number of connections of point  $x \in \Phi$  in this dominating RCG is

$$\begin{aligned}
\mathbb{E}^x \left[ \sum_{y \neq x \in \Phi} \mathbb{P}(M_{xy}(J) > 0 \mid \Psi) \right] &\leq \mathbb{E}^x \left[ \sum_{y \neq x \in \Phi} \mathbb{E}(M_{xy}(J) \mid \Psi) \right] \\
&= \mathbb{E}^x \left[ \sum_{y \neq x \in \Phi} l(J) f(\|x - y\|) \right] \\
&= \lambda l(J) \int_{\mathbb{R}^d} f(\|z\|) dz = \lambda l(J) a.
\end{aligned}$$

Here,  $\mathbb{E}^x$  refers to the Palm probability of  $\Phi$  at  $x$ ; the second bound uses the fact that the probability that a non negative integer valued random variable is positive is less than its mean; the last relation follows from Slinsky's theorem and Campbell's formula. Hence, if the Lebesgue measure  $l(J)$  of  $J$  is small enough, there is hence no percolation in this dominating RCG [8, 9]. As a result, there is no percolation in the initial model.

This last property immediately implies that for all  $p_1$  with  $x_{p_1} = x$  and all non-increasing sequences  $T_{p_1, p_2}, T_{p_2, p_3}, \dots$  with  $T_{p_1, p_2} = t$ , we have  $T_{p_k, p_{k+1}} < t - \epsilon$  for all  $k$  larger than some random but finite  $K$ . This then proves the result of the lemma by a finite induction.  $\square$

An important property which remains to be proved is that the result of the Sheriff algorithm does not depend of the ordering of points that it relies upon. This is the object of the following:

**Theorem 1.** *Assume that  $t_0 > -\infty$  and that Assumptions 0–1 hold. Then almost surely, for every point  $p$  born in  $(t_0, t_1)$ , with  $t_1 < \infty$ , the Sheriff either determines a unique death time  $d_p \leq t_1$  and the killer, or finds out that  $p$  is alive at time  $t_1$ . The result is independent of the order in which the points were enumerated. This uniquely defines the point process  $\Phi_t$  of nodes alive at time  $t$  for all  $t > 0$ .*

*Proof.* Consider Algorithm 1. Make Step 2, and consider the set  $P$  of all points whose stacks are looked at before the investigation stack is emptied again. By Lemma 1, all stacks contain, a.s., only a finite number of cards with  $T_{pq} < t_1$ , and all times  $T_{pq}$  are a.s. distinct. If  $P$  is infinite, it contains a sequence with the property appearing in Lemma 2. Thus, with probability one, the investigation stack empties. Repeating the cycle, every point, sooner or later and almost surely, either gets a death certificate or has all connection times in its stack larger than  $t_1$ , in which case it is alive at  $t_1$ .

It remains to show that, almost surely, the resulting configuration does not depend on the pre-defined order in which the points are investigated. Let us consider one realization of the triple  $(\Psi_{(t_0, t_1)}, \{T_{pq}\}, \{I_{pq}\})$  and two different numberings of the points, say  $\{p^{(1)}\}$  and  $\{p^{(2)}\}$ . Since every point gets, a.s., a death certificate in both processes, we only need to show that these certificates are a.s. identical in both processes. Assume that for some point  $p_1$  we have  $d_{p_1}^{(1)} > d_{p_1}^{(2)}$ , where the superscripts refer to the two orderings of points. There exists a  $p_2$  such that  $d_{p_1}^{(2)} = T_{p_1 p_2}$ . Since the card  $(p_1, p_2, T_{p_1 p_2})$  is present in process 1 and since  $p_1$  is alive after time  $T_{p_1 p_2}$  in process 1, it must not be killed by  $p_2$  at  $T_{p_1 p_2}$  in process 1 (keeping in mind that  $d_{p_1}^{(2)} = T_{p_1 p_2}$  implies  $I_{p_1 p_2} = 1$ ). Hence, it must be that  $p_2$  is already dead at that time, i.e.

$$d_{p_2}^{(1)} < T_{p_1 p_2} = d_{p_1}^{(2)}.$$

Consider now process 2. The fact that  $d_{p_1}^{(2)} = T_{p_1 p_2}$  implies that

$$T_{p_1 p_2} = d_{p_1}^{(2)} < d_{p_2}^{(2)}.$$

Hence, in view of  $d_{p_2}^{(1)} < d_{p_1}^{(2)}$ , we have  $d_{p_2}^{(2)} > d_{p_2}^{(1)}$ . Now, this reasoning can be continued, leading to an infinite sequence of distinct points  $p_n$  such that

$$d_{p_1}^{(1)} > d_{p_1}^{(2)} > d_{p_2}^{(1)} > d_{p_2}^{(2)} > d_{p_3}^{(1)} > d_{p_3}^{(2)} > \dots$$

and within this sequence ( $i$  alternating between 1 and 2)  $d_{p_n}^{(i)} = T_{p_n p_{n+1}}$  for  $n = 1, 2, \dots$ . But this sequence is exactly of the kind whose existence is a.s. denied by Lemma 2.  $\square$

Below we will take  $t_0 = 0$ . The Sheriff algorithm can be seen as a measurable mapping from  $(M(\mathbb{R}^d \times \mathbb{R}), (0, \infty)^{\mathbb{N}}, \{0, 1\}^{\mathbb{N}})$  to  $M(\mathbb{R}^d \times \mathbb{R})$  with

$$\text{Sheriff}(\Psi_{(0, t_1)}, \{T_{pq}\}, \{I_{pq}\}) := \{x_p, d_p\}_{p \in \Psi_{(0, t_1)}}, \quad (6)$$

where, in the case  $t_1 < \infty$ , we set  $d_p = \infty$  for points living at time  $t_1$ .

### 3.2 More General Initial Conditions

The initial condition of the Sheriff algorithm was empty at time  $t_0$  since the stacks were defined from the arrivals in  $(t_0, t_1)$ . It will be useful below to extend this to an initial condition made of a point process  $\mathcal{Z}_0$  of nodes already present (i.e. born) at time  $t_0$  and having independent pairwise random connections and killing direction variables as those defined above. In

this case, an initial stack is built for each node of  $\mathcal{Z}_0 \cup \Psi_{(t_0, t_1)}$ , containing its sorted sentences. If the point process  $\mathcal{Z}_0$  satisfies the property in Lemmas 1 and 2 namely if

1. for every  $z$  in  $\mathcal{Z}_0$ , the set of all  $T_{z,w}, w \in \mathcal{Z}_0$  which belong to  $(t_0, t)$  is finite for  $-\infty < t_0 < t < \infty$ ;
2. there is no infinite sequence of points  $z_1, z_2, \dots$  of  $\mathcal{Z}_0$  such that the sequence  $T_{z_n z_{n+1}}$  be non-increasing,

then there is no difficulty in running Sheriff on this initial condition.

**Throughout the paper, the initial condition will be assumed to be a motion invariant point process [4] satisfying the conditions 1 and 2 given above.**

Here are a few examples where this condition is satisfied. If  $\mathcal{Z}_0$  is Poisson, homogeneous and independent of  $\Psi_{(t_0, t_1)}$ , this follows from Lemma 2. By a direct monotonicity argument, the same holds if  $\mathcal{Z}_0$  is any compatible thinning<sup>5</sup> of an independent homogeneous Poisson point process. This compatible thinning can be based on the independent pairwise connections and killing directions. In particular let  $\Phi_{t_1}$  denote the point process, built by Sheriff, of nodes living at time  $t_1$  when the system starts empty at time  $t_0$ . This is a motion invariant thinning of  $\Psi_{(t_0, t_1)}$  based on these pairwise variables. One can hence apply Sheriff on  $[t_1, t_2)$  to the initial condition  $\mathcal{Z}_0 = \Phi_{t_1}$  for all  $t_1 < t_2 < \infty$ .

### 3.3 The Double Card Version of Sheriff

The Sheriff algorithm could also be defined as follows: in the initialization, for each connection time  $T_{pq}$ , put a card  $(p, q, T_{pq})$  in  $p$ 's stack and a card  $(q, p, T_{pq})$  in  $q$ 's stack (remember that  $T_{pq} = T_{qp}$ ). The values of the  $I_{pq}$ 's are not drawn beforehand, so that we don't speak of death sentences but of *duel times*. In Step 2 of Sheriff, copy (instead of move) the card of the next point whose top card carries a time less than  $t_1$  and is not a death certificate to the investigation stack. In step 3, there are three alternatives: (i) if  $q$ 's stack has a death certificate on top, then  $p$ 's stack and the investigation stack are popped; (ii) if  $q$ 's top card carries a duel time less than  $T_{pq}$ , that card is copied to the investigation stack; (iii) in the remaining case,  $q$ 's top card

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<sup>5</sup>By compatible thinning, we mean a thinning where the retention decisions are marks of the point process.

is  $(q, p, T_{qp})$ ; now the  $I_{pq}$  variable is drawn, the loser's top card is replaced by a death certificate, and the killer's stack and the investigation stack are popped. It is obvious that this variant, which will be referred to as the *double card* version of the algorithm, performs similarly to the initial one (although the investigation order is not exactly the same).

The double-card version makes it clear that the direction of the interaction (i.e. the value of  $I_{pq}$ ) need not be specified before the step when it is really needed in the algorithm at the realization of a duel. Another nice feature is that full information on the  $\{T_{pq}\}$  sequence remains in the stacks. Indeed, for an unrealized duel, which can only happen when either or both duelists are dead before the execution time, a copy of the card remains in the stack of at least one of the two duelists.

## 4 Initial Condition and Coupling

In this section, we investigate how additions to the initial condition perturb the history of all other points. This is done through a coupling, called Sheriff<sup>Z</sup>, which allows one to jointly build the histories with and without these additions. In Section 7, we will leverage the finiteness of this perturbation to construct the steady state through a coupling from the past.

### 4.1 Augmenting the Initial Condition

Below, we consider two systems: (1) that with an empty set of nodes as initial condition (as in Sheriff); (2) that with an initial condition consisting of a point process  $\mathcal{Z}_0$  in  $\mathbb{R}^d \times \{0\}$ , satisfying the conditions of Subsection 3.2 and representing some additional set of points born at time 0. The point process  $\mathcal{Z}_0$  will be called the *augmentation* point process. The first case is a special case of the second one (with  $\mathcal{Z}_0 = \emptyset$ ) and will be referred to as the *non-augmented* case.

Our aim below will be to jointly build two parallel executions of the killing history: that with this augmentation and that without. The coupling consists of using the same sequences of connections ( $\{T_{pq}\}$ ) for common points (the points of  $\Psi_{(0,t_1)}$ ). The addition of  $\mathcal{Z}_0$  has non-monotonic effects on the life times of common points, with some points having their lifetime extended and others shortened. In the algorithm described below, at any given time, we call *zombies* the points that are alive in the augmented process and are dead in the non augmented process (i.e. with a death time already determined in the non-augmented process and not yet determined in the

augmented one)<sup>6</sup>. Conversely we will call *antizombies* the points that are dead in the augmented process and alive in the original process (i.e. with a death time already determined in the augmented process and not yet in the non-augmented one).

At any given time, zombies and antizombies will be called *special points* and the other points will be called *regular*. The basic principles of the joint execution are as follows:

- The killing of a regular point by another regular point determines the death times of the former in the two processes (these death times are equal).
- If a zombie kills a regular point, this determines the death time of the latter in the augmented process. This regular point becomes an anti-zombie and is kept in the algorithm until its death time is determined in the non-augmented process.
- If an antizombie kills a regular point, this determines the death time of the latter in the non-augmented process. This point becomes a zombie and is kept in the algorithm until its death time is determined in the augmented process.
- If a regular point kills a zombie, this determines the death time of the latter in the augmented process, and this zombie can be forgotten as its two death times are now determined in both processes.
- If a regular point kills an antizombie, this determines the death time of the latter in the non-augmented process, and the antizombie can be forgotten for the same reasons as above.
- If a zombie (resp. an antizombie) kills another zombie (resp. antizombie), this determines the death time of the latter in the augmented (resp. non-augmented) process and the killed point can be forgotten.
- Zombies and antizombies cannot kill each other as they belong to different processes.

See Figure 1 for an illustration.

The Sheriff<sup>Z</sup> algorithm described below generates the announced coupling of the original and the augmented process. It simultaneously builds

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<sup>6</sup>It makes sense to call the points of  $\mathcal{Z}_0$  zombies as well.

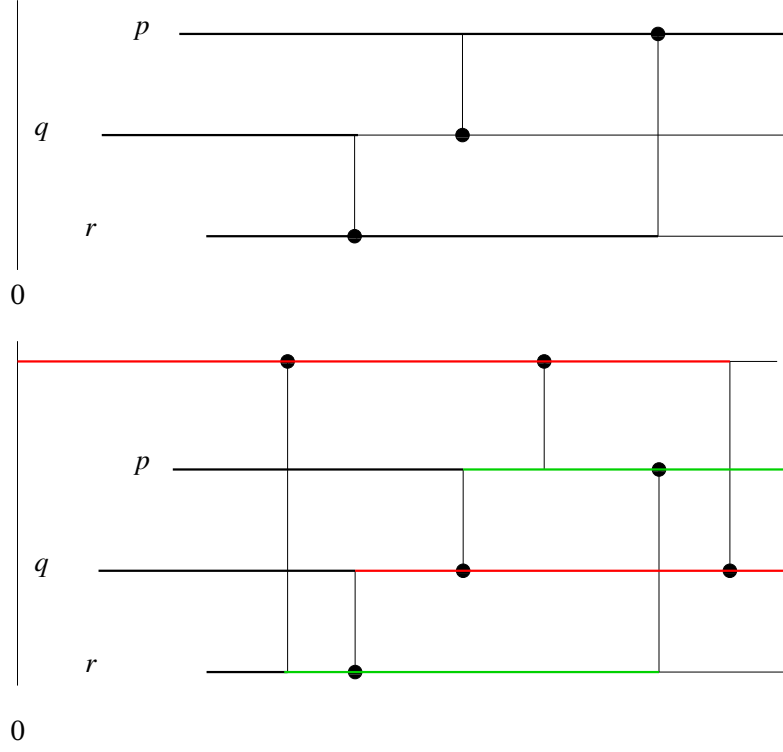


Figure 1: Two coupled killing processes. Time is on the  $x$  axis; space on the  $y$  axis. Dots associated with vertical arrows indicate killers. Upper figure: a killing process with three points. Lower figure: same process with an added point born at time zero. Colors: *black*: regular point; *red*: zombie; *green*: antizombie.

two sequences  $\{e_p\}_{p \in \Psi_{(0,t_1)}}$  and  $\{e'_p\}_{p \in \mathcal{Z}_0 \cup \Psi_{(0,t_1)}}$  that will be shown later to coincide with the death sequences of the non-augmented and the augmented systems, respectively. It does so by maintaining the list of zombies and antizombies at all times.

**Algorithm 2.** Sheriff<sup>Z</sup> (“Sheriff, with zombies”).

Input:  $\mathcal{Z}_0$ ,  $\Psi_{(0,t_1)}$ ,  $\{T_{pq}\}_{p,q \in \mathcal{Z}_0 \cup \Psi_{(0,t_1)}}$ ,  $\{I_{pq}\}_{p,q \in \mathcal{Z}_0 \cup \Psi_{(0,t_1)}}$ .

Output:  $\{e_p \in [0, t_1]\}_{p \in \Psi_{(0,t_1)}}$ ,  $\{e'_p \in [0, t_1]\}_{p \in \mathcal{Z}_0 \cup \Psi_{(0,t_1)}}$ .

1. Initialization:

- For each  $p \in \mathcal{Z}_0 \cup \Psi_{(0,t_1)}$ , build a stack  $S_p$  of cards of the form

$(p, q, T_{pq})$ , where the  $T_{pq}$  variables are sorted in increasing order (earliest time on top)<sup>7</sup>;

- The sheriff has an investigation stack (IS), initially empty.
  - For all  $p$ ,  $e_p := e'_p := \infty$ ;
  - For all  $z \in \mathcal{Z}_0$ , one maintains the point sets  $\mathcal{A}(z)$  (“antizombies”) and  $\mathcal{Z}(z)$  (“zombies”) offspring of  $z$ ; initially,  $\mathcal{A}(z) := \emptyset$  and  $\mathcal{Z}(z) := \{z\}$ ; if at some time  $p \in \mathcal{Z}(z)$  or  $p \in \mathcal{A}(z)$ , we define  $\mathfrak{z}(p) = z$  (the value will be uniquely defined); denote (at all times)  $\mathcal{A} = \cup_{z \in \mathcal{Z}_0} \mathcal{A}(z)$ ,  $\mathcal{Z} = \cup_{z \in \mathcal{Z}_0} \mathcal{Z}(z)$ ;
  - Call a point  $p$  *finished*, if either  $p \in \mathcal{Z}_0$  and  $e'_p < \infty$ , or  $p \in \Psi_{(0, t_1)}$  and  $e_p \vee e'_p < \infty$ , or  $S_p$ ’s top card has  $T_{pq} \geq t_1$ .
2. If IS is empty, the sheriff chooses, from a pre-defined ordering of all points, the first unfinished point  $p$  such that the top card of  $S_p$  has  $T_{pq} < t_1$ , and copies this card to IS.
  3. The sheriff looks at the top card of IS, say  $(p, q, T_{pq})$ .
    - if  $q$  is finished, he pops both  $S_p$  and IS; goes to Step 2;
    - if  $S_q$ ’s top card has  $T_{qr} < T_{pq}$ , he copies this last card to IS; goes to Step 3.
  4. The sheriff does one of the following (if the appropriate case is missing, interchange  $p$  and  $q$ ):

$p, q \in \mathcal{A}^c \cap \mathcal{Z}^c$ :

- if  $I_{pq} = 1$ ,  $e_p := e'_p := T_{pq}$ ; pops  $S_q$ ;
- if  $I_{pq} = 0$ ,  $e_q := e'_q := T_{pq}$ ; pops  $S_p$ ;

$p \in \mathcal{Z}$  and  $q \in \mathcal{A}^c \cap \mathcal{Z}^c$ :

- if  $I_{pq} = 1$ ,  $e'_p := T_{pq}$ ;  $\mathcal{Z}(\mathfrak{z}(p)) := \mathcal{Z}(\mathfrak{z}(p)) \setminus \{p\}$ ; pops  $S_q$ ;
- if  $I_{pq} = 0$ ,  $e'_q := T_{pq}$ ;  $\mathcal{A}(\mathfrak{z}(p)) := \mathcal{A}(\mathfrak{z}(p)) \cup \{q\}$ ; pops  $S_p$  and  $S_q$ ;

$p \in \mathcal{A}$  and  $q \in \mathcal{A}^c \cap \mathcal{Z}^c$ :

- if  $I_{pq} = 1$ ,  $e_p := T_{pq}$ ;  $\mathcal{A}(\mathfrak{z}(p)) := \mathcal{A}(\mathfrak{z}(p)) \setminus \{p\}$ ; pops  $S_q$ ;
- if  $I_{pq} = 0$ ,  $e_q := T_{pq}$ ;  $\mathcal{Z}(\mathfrak{z}(p)) := \mathcal{Z}(\mathfrak{z}(p)) \cup \{q\}$ ; pops  $S_p$  and  $S_q$ ;

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<sup>7</sup>We use here the double card version so that for all cards of the form  $(p, q, T_{pq})$  stored in  $p$ ’s stack, a card with the same data is also stored in  $q$ ’s stack.



$p, q \in \mathcal{Z}$ :

- if  $I_{pq} = 1$ ,  $e'_p := T_{pq}$ ;  $\mathcal{Z}(\mathfrak{z}(p)) := \mathcal{Z}(\mathfrak{z}(p)) \setminus \{p\}$ ; pops  $S_q$ ;
- if  $I_{pq} = 0$ ,  $e'_q := T_{pq}$ ;  $\mathcal{Z}(\mathfrak{z}(q)) := \mathcal{Z}(\mathfrak{z}(q)) \setminus \{q\}$ ; pops  $S_p$ ;

$p, q \in \mathcal{A}$ :

- if  $I_{pq} = 1$ ,  $e_p := T_{pq}$ ;  $\mathcal{A}(\mathfrak{z}(p)) := \mathcal{A}(\mathfrak{z}(p)) \setminus \{p\}$ ; pops  $S_q$ ;
- if  $I_{pq} = 0$ ,  $e_q := T_{pq}$ ;  $\mathcal{A}(\mathfrak{z}(q)) := \mathcal{A}(\mathfrak{z}(q)) \setminus \{q\}$ ; pops  $S_p$ ;

$p \in \mathcal{Z}$  and  $q \in \mathcal{A}$ :

- pops  $S_p$  and  $S_q$ .

5. The sheriff pops IS and goes to Step 2.

**Remark 2.** *If the set  $\mathcal{Z}_0$  is empty, Sheriff<sup>Z</sup> reduces to the double-card version of Sheriff (see Subsection 3.3); in this case, the first case in Step 4 is always met.*

**Remark 3.** *In the second bullet of cases 2 and 3 in Step 4, one discards the top cards of both  $p$  and  $q$  because  $q$  does not kill  $p$  but only labels it, and the connection between the two can be forgotten.*

## 4.2 Properties of the Sheriff<sup>Z</sup> Map

Let us now see in detail what Sheriff<sup>Z</sup> does. Let

$$\{x_p, d_p\}_{p \in \Psi_{(0, t_1)}} = \text{Sheriff}(\Psi_{(0, t_1)}, \{T_{pq}\}, \{I_{pq}\}),$$

where  $\{T_{pq}\}$  and  $\{I_{pq}\}$  are indexed by  $p, q \in \Psi_{(0, t_1)}$  and let

$$\{x, p, d'_p\}_{p \in \mathcal{Z}_0 \cup \Psi_{(0, t_1)}} = \text{Sheriff}(\mathcal{Z}_0 \cup \Psi_{(0, t_1)}, \{T_{pq}\}, \{I_{pq}\}),$$

where  $\{T_{pq}\}$  and  $\{I_{pq}\}$  are now indexed by  $p, q \in \mathcal{Z}_0 \cup \Psi_{(0, t_1)}$ . In these last definitions, we use the same sequences  $\{T_{pq}\}$  and  $\{I_{pq}\}$  as in Sheriff<sup>Z</sup>. Then we have:

**Theorem 2.** *Under Assumptions 0–1, the following claims hold almost surely:*

1. *The algorithm Sheriff<sup>Z</sup> runs unambiguously and every point gets finished (in the sense defined at the end of the initialization) in finite time. For all  $p \in \Psi_{(0, t_1)}$ ,  $e_p = d_p$ , whereas for all  $p \in \mathcal{Z}_0 \cup \Psi_{(0, t_1)}$ ,  $d'_p = e'_p$ .*

2. For each  $z \in \mathcal{Z}_0$ , the Sheriff<sup>Z</sup> algorithm (implicitly) generates the set-valued stochastic processes  $(A_t(z))_{t \in [0, t_1]}$  and  $(Z_t(z))_{t \in [0, t_1]}$  representing, respectively, the antizombies and zombies originating from  $z$  and living at time  $t$ . These sets satisfy the conditions

$$p \in A_t(z) \text{ for some } z \Leftrightarrow d'_p \leq t < d_p, \quad (7a)$$

$$p \in Z_t(z) \text{ for some } z \Leftrightarrow d_p \leq t < d'_p. \quad (7b)$$

The “families” of offsprings

$$O_{t_1}(z) = \bigcup_{t \in [0, t_1)} A_t(z) \cup \bigcup_{t \in [0, t_1)} Z_t(z)$$

of distinct  $z$ 's are disjoint.

*Proof.* First note that if, in any phase of the algorithm, a point  $p$  belongs to  $A(z) \cup Z(z)$  for some  $z \in \mathcal{Z}_0$ , then  $z$  is unique and we can thus denote it as  $\mathfrak{z}(p)$ . Indeed, this holds true for the initial situation where  $A(z) \cup Z(z) = \{z\}$ , and any given point can be added to some  $A(z) \cup Z(z)$  only once. Thus, the steps of the algorithm are unambiguously defined.

Second, note that we now use double cards as discussed in Subsection 3.3. All points get finished (a.s.) by the argumentation used for proving the same for Sheriff.

Since the algorithm clearly fixes the times when a point becomes or ceases to be a zombie or antizombie, it is obvious that the processes  $(A_t(z))_{t \in [0, t_1]}$  and  $(Z_t(z))_{t \in [0, t_1]}$  are well defined.

A further examination of the algorithm yields the conditions (7). We also see why zombies and antizombies don't interact (last case of step 4): if  $p \in A_t(z)$  and  $q \in Z_t(z')$ , at time  $t$ ,  $p$  is dead in the augmented scenario and  $q$  is dead in the original scenario. Claim 2 is now proven, since the last subclaim just states the uniqueness discussed already in the beginning of the proof.

To show that  $d_p = e_p$  for all  $p$ , note that zombies are points that already have the  $e$ -value set. When a zombie kills a regular point, the latter receives an  $e'$ -value and becomes an antizombie, but this has no effect on the setting of subsequent  $e_p$ -values. If zombies were considered as finished (that is, dead), the  $e_p$ -values would be the same.

This is illustrated in Figure 2.

Similarly, the antizombies have no effect on the setting of  $e_p$ -values — if we are interested only in the latter, antizombies could as well be considered as finished. This completes the proof of Claim 1.  $\square$



We define inductively a non-decreasing sequence of stopping times  $T_n$  and a sequence of random variables  $J_n \in \{-1, 0, 1\}$  by setting  $T_0 = J_0 = 0$  and

$$T_{n+1} = \begin{cases} \infty, & \text{if } T_n = \infty, \\ \inf \{T_{xy} : x \in U_{T_n}, y \in S_{T_n}(z)\}, & \text{otherwise,} \end{cases}$$

with  $\inf \emptyset = \infty$ . Assume that, for some  $n$ ,  $S_{T_n}$  is finite and non-empty. By Lemma 1 we have, a.s.,  $T_n < T_{n+1} < \infty$ , and  $S_t$  does not change on  $[T_n, T_{n+1})$ . Let now  $T_{n+1} = T_{xy}$ , where  $y \in S_{T_n}(z)$ , and if  $x \in S_{T_n}(z)$ , we choose for definiteness  $x$  as being farther from the origin than  $y$ . At time  $T_{n+1}$ , one of the following takes place:

- Case 0:  $x$  is finished before time  $T_{xy}$ , or  $x$  is, before  $T_{n+1}$ , a special point of the kind opposite to that of  $y$ ; then  $S_{T_{n+1}}(z) = S_{T_n}(z)$ .
- Case 1:  $x$  is a regular point. Then the chances, determined by  $I_{xy}$ , are  $\frac{1}{2}$  that  $S_{T_{n+1}}(z) = S_{T_n}(z) \cup \{x\}$  (with  $x$  being killed by a zombie or an antizombie and being transformed to the opposite kind) and  $\frac{1}{2}$  that  $S_{T_{n+1}}(z) = S_{T_n}(z) \setminus \{y\}$  ( $y$  being killed by  $x$ ).
- Case 2:  $x$  is a special point of the same kind as  $y$ , and  $x \notin S_{T_n}(z)$ . Now the chances are  $\frac{1}{2}, \frac{1}{2}$  that  $S_{T_{n+1}}(z) = S_{T_n}(z)$  or  $S_{T_{n+1}}(z) = S_{T_n}(z) \setminus \{y\}$ .
- Case 3:  $x$  and  $y$  are special points of same kind, and  $x \in S_{T_n}(z)$ . Then  $S_{T_{n+1}}(z) = S_{T_n}(z) \setminus \{x\}$  or  $S_{T_{n+1}}(z) = S_{T_n}(z) \setminus \{y\}$ , depending on  $I_{xy}$ .

In Case 0 we set  $J_{n+1} = 0$ , and in the remaining cases  $J_{n+1} = -1 + 2I_{xy}$ . On the other hand, if  $T_{n+1} = \infty$ , we set  $J_{n+1} = 0$ , and interpret  $S_{T_{n+1}}(z) = S_{T_n}(z)$ . Note that  $S_{T_{n+1}}$  is finite in every case, and

$$s_n := |S_{T_n}(z)| \leq 1 + \sum_{k=1}^n J_k, \quad n \geq 0. \quad (8)$$

In fact, the sequence  $s_n$  is an  $(\mathcal{F}_{T_n})$ -supermartingale. If  $T_n = \infty$  for some  $n$ , we clearly have  $|O_\infty(z)| < \infty$ . If  $T_n < \infty$  for all  $n$  but  $J_n \neq 0$  for only finitely many  $n$ , then  $O_{T_n}$  remains unchanged for  $n \geq n_0$ , and  $T_n \rightarrow \infty$  by Lemma 1, thus again  $|O_\infty(z)| < \infty$ . Finally, assume that we have with positive probability  $T_n < \infty$  for all  $n$  and  $J_n \neq 0$  for infinitely many  $n$ . Now, such  $J_n$ s are independent random variables taking values  $\pm 1$  with probabilities  $\frac{1}{2}, \frac{1}{2}$ . Since a symmetric random walk on  $\mathbb{N}$  hits zero with probability 1, there is a finite random number  $c$  such that  $1 + \sum_{k=1}^c J_k = 0$ . Now, by (8),  $s_c = 0$ , and we get  $T_{c+1} = \infty$ , which contradicts the assumption. This concludes the proof.  $\square$

### 4.3 Section Summary

Let us summarize this section by focusing on the case where the augmentation point process  $\mathcal{Z}_0$  is translation invariant in  $\mathbb{R}^d$ . We established the following results:

1. Theorem 2 uniquely defines the marked point process  $\tilde{\Phi}_t$  of nodes which are not finished at time  $t < \infty$  under Sheriff<sup>Z</sup>; the marks belong to the set  $\{\mathcal{R}, \mathcal{A}, \mathcal{Z}\}$ . The points with mark  $\mathcal{R}$  are regular points, which are alive both in the augmented and the non-augmented processes, whereas those with mark  $\mathcal{A}$  (resp.  $\mathcal{Z}$ ) are antizombies (resp. zombies) with a life shorter (resp. longer) in the augmented process compared to the non-augmented one.
2. Theorem 2 also shows that the points of  $\tilde{\Phi}_t$  with marks in  $\mathcal{R} \cup \mathcal{Z}$  form a stationary point process  $\Phi'_t$  which coincides with that built by Sheriff at time  $t$  when the initial condition is  $\mathcal{Z}_0$ . Similarly, the points of  $\tilde{\Phi}_t$  with marks in  $\mathcal{R} \cup \mathcal{A}$  form a stationary point process  $\Phi_t$  which coincides with that built by Sheriff at time  $t$  when the initial condition is  $\emptyset$ .
3. Theorem 3 shows that the set of special points of  $(\tilde{\Phi}_t)_{t \geq 0}$  which are offsprings of a given point  $z \in \mathcal{Z}_0$  has a finite cardinality a.s. This collection of sets is translation invariant.

## 5 Non Degeneracy of Transient Densities

From now on, the augmentation point process  $\mathcal{Z}_0$  is assumed to be motion invariant in  $\mathbb{R}^d$  and to satisfy the assumptions of Subsection 3.2.

### 5.1 Tightness

This section contains a simple stochastic comparison argument showing that the stochastic processes built by Sheriff are tight, which in turn implies that densities admit a uniform upper bound.

Let us define a *mutual-service process with parameters*  $(\tilde{\lambda}, \tilde{\mu})$  as the birth-death process whose birth and death intensities in state  $j$  are

$$\lambda_j \equiv \tilde{\lambda}, \quad \mu_j = j(j-1)\tilde{\mu}, \quad j = 0, 1, 2, \dots$$

Note that although a mutual-service process may start from state 0, it cannot reach 0 from any other state  $j > 0$ .

**Proposition 1.** *Under Assumptions 0–2, if  $\mathcal{Z}_0$  is a translation invariant thinning of a Poisson process on  $\mathbb{R}^d$ , then the process  $(\Phi_t)_{t \geq 0}$  built by Sheriff satisfies the following properties:*

1. *For any  $t > 0$ ,  $\Phi_t$  is spatially stationary and ergodic.*
2. *Let  $b > 0$  be sufficiently small to satisfy  $f(b\sqrt{d}) > 0$ ; tessellate  $\mathbb{R}^d$  into cubes  $C_i$  of side  $b$  indexed by  $i \in \mathbb{Z}^d$ , assuming that the center of  $C_0$  is the origin. Then there exists a mutual service process  $U(i)$  with parameters  $\tilde{\lambda} = \lambda b^d$ ,  $\tilde{\mu} = 2f(b\sqrt{d})$ , such that, a.s.,*

$$\Phi_t(C_i) \leq U_t(i), \quad t \geq 0, \quad i \in \mathbb{Z}^d, \quad (9)$$

*and the processes  $U(i)$  are independent given their initial states*

$$U_0(i) = \mathcal{Z}_0(C_i), \quad i \in \mathbb{Z}^d. \quad (10)$$

3. *The intensities  $\beta_{\Phi_t}$  satisfy the bound*

$$\beta_{\Phi_t} < c, \quad t \geq 0, \quad (11)$$

*with  $c$  a finite constant.*

4. *For all positive integers  $k$  and for all bounded Borel sets  $C$ ,  $\mathbb{E}(\Phi_t(C)^k)$  is uniformly bounded in  $t$ .*

*Proof.* Claim 1: the (space) stationarity and the ergodicity follow from the fact that the point process  $\Phi_t$  is a translation invariant thinning of an independently marked stationary and ergodic point process.

Claim 2: by a classical coupling argument, we can construct the processes  $U_t(i)$  on an extension of the probability space of  $(\Phi_t)$  so that (9) and the conditional independence hold. Set the initial states of  $U_t(i)$  according to (10). For each  $i \in \mathbb{Z}^d$ , we can obviously make the up-jumps of  $\Phi(C_i)$  and  $U_t(i)$  identical. For down-jumps, assume that  $\Phi_t \cap C_i = \{X_1, \dots, X_m\}$  with  $m \geq 2$ , and that  $U_t(i) = j \geq m$ . Given  $\Phi_t$ , the times  $T_{X_{i_1} X_{i_2}}$  are independent exponentially distributed random variables with parameters  $2f(\|X_{i_1} - X_{i_2}\|) \geq \tilde{\mu}$ , respectively. If  $j = m$ ,

$$\min_{i_1} d_{X_{i_1}} \leq \min_{i_1, i_2} T_{X_{i_1} X_{i_2}} \stackrel{(st)}{\leq} \text{Exp}(j(j-1)\tilde{\mu}),$$

with *st* denoting stochastic ordering. The claim then follows from these observations.

Claims 3 and 4 follow from Claim 2 since, except for state 0, a mutual-service process is dominated by an M/M/ $\infty$  queue with the same parameters, whose stationary distribution is Poisson.  $\square$

**Remark 4.** For all thinned Poisson initial conditions  $\mathcal{Z}_0$  (such initial conditions satisfy the above assumptions), one can adapt the last proof and obtain analogues of Proposition 1 for

1. The point process  $\Phi'_t = \Phi_{t, \mathcal{Z}_0}$  built by Sheriff;
2. The point process  $\tilde{\Phi}_t$  built by Sheriff<sup>Z</sup>; for showing this last property, one can use the fact that  $\tilde{\Phi}_t$  is bounded from above by the superposition of the point processes  $\Phi_t$  and  $\Phi'_t$ , which both satisfy the desired properties.

**Lemma 3.** Let  $\mathbb{E}_\chi^0$  stand for the Palm probability of a point process  $\chi$ . Under Assumptions 0–3, there exists a finite constant  $c$  such that

$$\beta_{\Phi_t} \mathbb{E}_{\Phi_t}^0 \sum_{X \in \Phi_t} f(\|X\|) \leq c, \quad \forall t \in \mathbb{R}. \quad (12)$$

More generally, for all positive integers  $k$ , there exists a finite constant  $c_k$  such that

$$\beta_{\Phi_t} \mathbb{E}_{\Phi_t}^0 \left[ \left( \sum_{X \in \Phi_t} f(\|X\|) \right)^k \right] \leq c_k, \quad \forall t \in \mathbb{R}. \quad (13)$$

*Proof.* Using the product form upper bound described above, we get

$$\begin{aligned} \beta_{\Phi_t} \mathbb{E}_{\Phi_t}^0 \sum_{X \in \Phi_t} f(\|X\|) &= \frac{1}{b^d} \mathbb{E} \sum_{X \in \Phi_t \cap C_0} \sum_{Y \neq X \in \Phi_t} f(\|X - Y\|) \\ &\leq \frac{K}{b^d} \mathbb{E} \Phi_t(C_0)^2 \\ &\quad + \frac{1}{b^d} \mathbb{E} \sum_{X \in \Phi_t \cap C_0} \sum_{i \neq 0} \sum_{Y \in \Phi_t \cap C_i} f(\|X - Y\|) \\ &\leq \frac{K}{b^d} \mathbb{E} \Phi_t(C_0)^2 + \frac{(\mathbb{E} U_t(0))^2}{b^d} \sum_{i \neq 0} f(d_i+), \end{aligned}$$

where  $K$  is the upper-bound on  $f$  (Assumption 3),  $U(0)$  is the mutual service process defined in the proof of Proposition 1,  $d_i$  is the distance from  $C_i$  to  $C_0$ , and  $d_i+$  stands for the right-hand limit (to handle the case where  $d_i = 0$ ). There exists a constant  $H > 1$  and a ball  $B$  centered in the origin such that for all  $i$  with  $C_i$  not included in  $B$  and for all  $x \in C_i$ ,  $\|x\| \leq H d_i$ . If  $\nu_d$  denotes the volume of a unit ball, this in turn implies that

$$\frac{1}{b^d} \sum_{i \neq 0, C_i \notin B} f(d_i) \leq \int_{\mathbb{R}^d} f\left(\frac{\|x\|}{H}\right) dx = d \nu_d \int_{r>0} f\left(\frac{r}{H}\right) r^{d-1} dr = H^d a < \infty.$$

The proof of the first statement is then concluded from the second statement of Proposition 1 and from the fact that the moments of  $U_t(0)$  are uniformly bounded.

For  $k \geq 1$ , using again the product form upper bound, we get

$$\begin{aligned} \beta_{\Phi_t} \mathbb{E}_{\Phi_t}^0 \left( \sum_{X \in \Phi_t} f(\|X\|) \right)^k &= \frac{1}{b^d} \mathbb{E} \sum_{X \in \Phi_t \cap C_0} \left( \sum_{Y \neq X \in \Phi_t} f(\|X - Y\|) \right)^k \\ &\leq \frac{K^k}{b^d} \mathbb{E}(\Phi_t(C_0))^k \\ &\quad + \frac{1}{b^d} \mathbb{E}U_t(0) \sum_{n_i \geq 0: \sum_{i \neq 0} n_i = k} \prod_{i \neq 0: n_i > 0} f^{n_i}(d_i) \mathbb{E}\Phi_t(C_i)^{n_i}. \end{aligned}$$

Using now the fact that there exists a constant  $J \geq 1$  such that

$$\mathbb{E}\Phi_t(C_i)^n \leq J(\mathbb{E}\Phi_t(C_i))^n$$

uniformly in  $i$ ,  $t$  and  $n \leq k$  (Proposition 1), we get that

$$\begin{aligned} \beta_{\Phi_t} \mathbb{E}_{\Phi_t}^0 \left( \sum_{X \in \Phi_t} f(\|X\|) \right)^k &\leq \frac{K^k}{b^d} \mathbb{E}(\Phi_t(C_0))^k \\ &\quad + \frac{J}{b^d} (\mathbb{E}U_t(0))^{k+1} \left( \sum_{i \neq 0} f(d_i) \right)^k \end{aligned}$$

and the proof of the second assertion then follows from the finiteness of  $a$  as above.  $\square$

The results of the last lemma extend to the point process  $\Phi'_t = \Phi_{t, \mathcal{Z}_0}$  built by Sheriff for all thinned Poisson initial conditions.

## 5.2 Positiveness

We now prove that for all finite  $t$ , the densities of all our point processes are positive. We denote by  $\mathcal{R}_t$  (resp.  $\mathcal{Z}_t$ ,  $\mathcal{A}_t$  and  $\mathcal{S}_t$ ) the stationary point process of regular points (resp. zombies, antizombies and special points) built by Sheriff<sup>Z</sup> at time  $t$ . For each of these point processes, say  $\mathcal{X}_t$ , we denote its intensity by  $\beta_{\mathcal{X}_t}$ .

**Lemma 4.** *Make Assumptions 0-2, and let  $\beta_{\mathcal{Z}_0} > 0$ . Then  $\beta_{\mathcal{X}_t} > 0$  for all finite  $t$ , with  $\mathcal{X}_t = \mathcal{R}_t, \mathcal{Z}_t, \mathcal{A}_t, \mathcal{S}_t$ .*



*Proof.* Let  $t$  be fixed and finite. We start with  $\mathcal{X}_t = \mathcal{Z}_t$ . Let  $z$  be a typical point of  $\mathcal{Z}_0$ . The total number of points of  $\Psi(0, t)$  that have a connection to  $z$  that takes place before time  $t$  is (stochastically) bounded from above by a Poisson random variable with parameter

$$\int_{\mathbb{R}^d} (1 - e^{-f(\|x-z\|)}) \lambda t dx \leq \lambda t a.$$

The total number of points of  $\mathcal{Z}_0$  that have a connection to  $z$  is also finite by assumption (Item 2 in Subsection 3.2). Hence the probability that all duels involving  $z$  and taking place before time  $t$  are oriented in such a way that  $z$  survives is positive. This shows that  $\beta_{\mathcal{Z}_t} > 0$  and also that  $\beta_{\mathcal{S}_t} > 0$ .

In order to prove the result for  $\mathcal{X}_t = \mathcal{A}_t$ , we pick  $\epsilon < t$  and we use arguments similar to those above to show that the probability that (1)  $z$  survives until time  $t - \epsilon$ ; (2)  $\Psi(t - \epsilon, t)$  brings one arrival which kills  $z$  (which becomes an antizombie); and (3) the latter survives until time  $t$ , is positive.

In order to prove the result for  $\mathcal{X}_t = \mathcal{R}_t$ , we pick  $\epsilon < t$  and we look at the arrivals of  $\Psi(0, \epsilon)$  in each box  $C_i = i + [0, 1)^d$ , where  $k$  ranges over  $\mathbb{Z}^d$ . For those boxes that have at least one arrival, pick the first of them. This defines a point process. For a typical point of this point process, say  $r$ , we use arguments similar to those above to show that the probability that  $r$  survives until time  $t$  is positive. This shows that  $\beta_{\mathcal{R}_t} > 0$ .  $\square$

**Remark 5.** *It follows from the last lemma and from 2. in Section 4.3 that the densities  $\beta_{\Phi_t}$  and  $\beta_{\Phi'_t}$  are also positive for all finite  $t$ .*

## 6 Differential Equations for Transient Moment Measures

The setting of this section is the same as that of Section 4, with the empty and augmented initial conditions. We complement the result on the finiteness of the special points stemming from a single point (Theorem 3) by a set of differential equations on the densities and higher order moment measures of nodes of all types. These equations will be needed in the coupling from the past arguments of the next section.

We assume that the augmentation  $\mathcal{Z}_0$  is a motion invariant point process satisfying the assumptions of Subsection 3.2. The default setting is that Assumptions 0–3 hold.

## 6.1 Densities

### 6.1.1 Sheriff

Let  $\beta_{\Phi'_t}$  denote the density of the point process  $\Phi'_t$  built by Sheriff for the initial condition  $\mathcal{Z}_0$ . Let  $\mathbb{E}_{\Phi'_t}^0$  denote its Palm probability (since  $\beta_{\Phi'_t} > 0$ , see Remark 5, the latter is well defined). For all  $x \in \mathbb{R}^d$ , let

$$\pi_{\Phi'_t}(x) = \sum_{X \in \Phi'_t} f(\|X - x\|). \quad (14)$$

This quantity can intuitively be interpreted as the *death pressure* exerted by  $\Phi'_t$  at  $x$ . The death rate of a typical node living at time  $t$  is  $\mathbb{E}_{\Phi'_t}^0 \pi_{\Phi'_t}(0)$ . The following equation is proved in Appendix 9.1.

$$\frac{d}{dt} \beta_{\Phi'_t} = \lambda - \beta_{\Phi'_t} \mathbb{E}_{\Phi'_t}^0 \pi_{\Phi'_t}(0). \quad (15)$$

From Proposition 1 and Property 1 at the end of Section 5.1, the term  $\beta_{\Phi'_t} \mathbb{E}_{\Phi'_t}^0 \pi_{\Phi'_t}(0)$  that we find on the R.H.S. of this differential equation is uniformly bounded in  $t$ .

From the fact that  $\beta_{\Phi'_t}$  is uniformly bounded and from (15), we also get that

$$\frac{1}{t} \int_0^t \beta_{\Phi'_u} \mathbb{E}_{\Phi'_u}^0 \pi_{\Phi'_u}(0) du = \lambda + o(1) \quad (16)$$

as  $t$  tends to infinity.

### 6.1.2 Sheriff<sup>Z</sup>

For all  $t > 0$ , for each of the point processes  $\mathcal{X}_t = \mathcal{R}_t, \mathcal{Z}_t, \mathcal{A}_t$  or  $\mathcal{S}_t$ , since  $\beta_{\mathcal{X}_t} > 0$  (Lemma 4), the Palm probability  $\mathbb{E}_{\mathcal{X}_t}^0$  w.r.t.  $\mathcal{X}_t$  is well defined.

Assuming  $\mathcal{X}_t$  is the point process of nodes that interact with a node located at  $x$ , we define the death pressure exerted by the nodes of  $\mathcal{X}_t$  on  $x$  as

$$\pi_{\mathcal{X}_t}(x) := \sum_{X \in \mathcal{X}_t} f(\|X - x\|), \quad \pi_{\mathcal{X}_t} := \pi_{\mathcal{X}_t}(0). \quad (17)$$

Since zombies and antizombies do not interact, we refine this definition in

the case of  $\mathcal{S}_t$  as follows:

$$\pi_{\mathcal{S}_t}(x) = \begin{cases} \sum_{y \in \mathcal{Z}_t} f(|x - y|) & \text{if } x \in \mathcal{Z}_t, \\ \sum_{y \in \mathcal{A}_t} f(|x - y|) & \text{if } x \in \mathcal{A}_t, \\ \sum_{y \in \mathcal{S}_t} f(|x - y|) & \text{otherwise.} \end{cases} \quad (18)$$

This must be taken into account when working with the Palm probability of special points, since the point at origin may be of either type. Consequently, the general relation

$$\mathbb{E}_{\mathcal{S}_t}^0 = \frac{\beta_{\mathcal{Z}_t}}{\beta_{\mathcal{S}_t}} \mathbb{E}_{\mathcal{Z}_t}^0 + \frac{\beta_{\mathcal{A}_t}}{\beta_{\mathcal{S}_t}} \mathbb{E}_{\mathcal{A}_t}^0 \quad (19)$$

gives, for example,

$$\mathbb{E}_{\mathcal{S}_t}^0 \pi_{\mathcal{S}_t} = \frac{\beta_{\mathcal{Z}_t}}{\beta_{\mathcal{S}_t}} \mathbb{E}_{\mathcal{Z}_t}^0 \pi_{\mathcal{Z}_t} + \frac{\beta_{\mathcal{A}_t}}{\beta_{\mathcal{S}_t}} \mathbb{E}_{\mathcal{A}_t}^0 \pi_{\mathcal{A}_t}.$$

Notice that the mass transport principle (see [10] or Appendix 9.2) implies that

$$\beta_{\mathcal{Z}_t} \mathbb{E}_{\mathcal{Z}_t}^0 \pi_{\mathcal{R}_t} = \beta_{\mathcal{R}_t} \mathbb{E}_{\mathcal{R}_t}^0 \pi_{\mathcal{Z}_t} \quad (20)$$

$$\beta_{\mathcal{A}_t} \mathbb{E}_{\mathcal{A}_t}^0 \pi_{\mathcal{R}_t} = \beta_{\mathcal{R}_t} \mathbb{E}_{\mathcal{R}_t}^0 \pi_{\mathcal{A}_t}. \quad (21)$$

**Lemma 5.** *Under the foregoing assumptions,*

$$\frac{d}{dt} \beta_{\mathcal{Z}_t} = -\beta_{\mathcal{Z}_t} \mathbb{E}_{\mathcal{Z}_t}^0 \pi_{\mathcal{Z}_t + \mathcal{R}_t} + \beta_{\mathcal{R}_t} \mathbb{E}_{\mathcal{R}_t}^0 \pi_{\mathcal{A}_t} \quad (22)$$

$$\frac{d}{dt} \beta_{\mathcal{A}_t} = -\beta_{\mathcal{A}_t} \mathbb{E}_{\mathcal{A}_t}^0 \pi_{\mathcal{A}_t + \mathcal{R}_t} + \beta_{\mathcal{R}_t} \mathbb{E}_{\mathcal{R}_t}^0 \pi_{\mathcal{Z}_t} \quad (23)$$

$$\frac{d}{dt} \beta_{\mathcal{R}_t} = \lambda - \beta_{\mathcal{R}_t} \mathbb{E}_{\mathcal{R}_t}^0 \pi_{\mathcal{R}_t + \mathcal{Z}_t + \mathcal{A}_t}, \quad (24)$$

where all the terms found on the right hand sides of these differential equations are uniformly bounded in  $t$ .

*Proof.* By arguments similar to those of Lemma 3, both

$$\beta_{\mathcal{R}_t} \mathbb{E}_{\mathcal{R}_t}^0 \pi_{\mathcal{R}_t + \mathcal{Z}_t} + \beta_{\mathcal{Z}_t} \mathbb{E}_{\mathcal{Z}_t}^0 \pi_{\mathcal{R}_t + \mathcal{Z}_t}$$

and

$$\beta_{\mathcal{R}_t} \mathbb{E}_{\mathcal{R}_t}^0 \pi_{\mathcal{R}_t + \mathcal{A}_t} + \beta_{\mathcal{A}_t} \mathbb{E}_{\mathcal{A}_t}^0 \pi_{\mathcal{R}_t + \mathcal{A}_t}$$

are uniformly bounded in  $t$ , which in turn implies that all the terms found on the right hand sides of the differential equations are uniformly bounded in  $t$ .

The death rate of a typical zombie is  $\mathbb{E}_{\mathcal{Z}_t}^0 \pi_{\mathcal{Z}_t + \mathcal{R}_t}$ . The rate at which a regular point is transformed into a zombie is  $\mathbb{E}_{\mathcal{R}_t}^0 \pi_{\mathcal{A}_t}$ . The equations are then obtained by arguments similar to those used in Appendix 9.1 to prove (15).  $\square$

Notice that (24) can be rewritten as

$$\frac{d}{dt} \beta_{\mathcal{R}_t} = \lambda - \beta_{\mathcal{R}_t} \mathbb{E}_{\mathcal{R}_t}^0 \pi_{\mathcal{R}_t} - \beta_{\mathcal{Z}_t} \mathbb{E}_{\mathcal{Z}_t}^0 \pi_{\mathcal{R}_t} - \beta_{\mathcal{A}_t} \mathbb{E}_{\mathcal{A}_t}^0 \pi_{\mathcal{R}_t}. \quad (25)$$

These equations are consistent with those established in the last subsection. When adding (22) and (24), and when using the fact that  $\beta_{\Phi'_t} = \beta_{\mathcal{Z}_t} + \beta_{\mathcal{R}_t}$ , we get

$$\begin{aligned} \frac{d}{dt} \beta_{\Phi'_t} &= \lambda - \beta_{\mathcal{R}_t} \mathbb{E}_{\mathcal{R}_t}^0 \pi_{\mathcal{R}_t} - \beta_{\mathcal{Z}_t} \mathbb{E}_{\mathcal{Z}_t}^0 \pi_{\mathcal{Z}_t} - 2\beta_{\mathcal{R}_t} \mathbb{E}_{\mathcal{R}_t}^0 \pi_{\mathcal{Z}_t} \\ &= \lambda - \beta_{\Phi'_t} \mathbb{E}_{\Phi'_t}^0 \pi_{\Phi'_t}, \end{aligned} \quad (26)$$

which is (15).

### 6.1.3 Properties of Densities

When adding (22) and (23) and when using the relation (19), we get:

**Lemma 6.** *Under the foregoing assumptions,*

$$\frac{d}{dt} \beta_{\mathcal{S}_t} = -\beta_{\mathcal{S}_t} \mathbb{E}_{\mathcal{S}_t}^0 \pi_{\mathcal{S}_t}. \quad (27)$$

Hence

$$\beta_{\mathcal{S}_t} = \beta_{\mathcal{S}_0} \exp \left( - \int_0^t \mathbb{E}_{\mathcal{S}_u}^0 \pi_{\mathcal{S}_u} du \right). \quad (28)$$

It follows from (27) that  $\beta_{\mathcal{S}_t}$  decreases and hence tends to a limit as  $t$  tends to  $\infty$ .

## 6.2 Death Pressure

We recall that  $\Phi'_t$  denotes the point process built by Sheriff for the initial condition  $\mathcal{Z}_0$ .

**Lemma 7.** *We have*

$$\frac{d}{dt} \left( \beta_{\Phi'_t} \mathbb{E}_{\Phi'_t}^0 \pi_{\Phi'_t} \right) = 2\lambda a \beta_{\Phi'_t} - 2\beta_{\Phi'_t} \mathbb{E}_{\Phi'_t}^0 \pi_{\Phi'_t}^2. \quad (29)$$

*Proof.* For all Borel sets  $C$ , let

$$\Pi_{\Phi'_t}(\Phi'_t \cap C) = \sum_{X \in \Phi'_t \cap C} \pi_{\Phi'_t}(X), \quad (30)$$

with  $\pi_{\Phi'_t}(\cdot)$  defined in (17). For all sets  $C$ , we have

$$\begin{aligned} \frac{d}{dt} \mathbb{E} \Pi_{\Phi'_t}(\Phi'_t \cap C) &= 2\lambda a \mathbb{E} |\Phi'_t \cap C| \\ &\quad - \mathbb{E} \sum_{X \in \Phi'_t \cap C} \pi_{\Phi'_t}^2(X) \\ &\quad - \mathbb{E} \sum_{Y \in \Phi'_t} \pi_{\Phi'_t \cap C}(Y) \pi_{\Phi'_t}(Y). \end{aligned} \quad (31)$$

The rationale is the following:  $\Pi_{\Phi'_t}(\Phi'_t \cap C)$  represents the pressure exerted by  $\Phi'_t$  on  $\Phi'_t \cap C$ . The reasons for this pressure to change with time are:

- A new point can be born anywhere from the Poisson rain process. For each  $X \in \Phi'_{t-} \cap C$ , the average pressure increase per time unit due to arrivals is  $\lambda a$ . In the case where that point is born in  $C$ , which happens with intensity  $\lambda|C|$ , it meets in average a pressure of strength  $\beta_{\Phi'_t} a$ , which is added to the total pressure. Using  $\beta_{\Phi'_t} |C| = \mathbb{E} |\Phi'_t \cap C|$ , the two effects give the first term of the R.H.S.
- Each  $X$  in  $\Phi'_t \cap C$  can be killed by another point. This happens with intensity  $\pi_{\Phi'_t}(X)$ . The death of  $X$  will decrease the total pressure by  $\pi_{\Phi'_t}(X)$ , hence the second term. This process also removes some pressure to the remaining points from  $\Phi'_t \cap C$ , but this effect is considered in the third term.
- Each  $Y$  in  $\Phi'_t$  can be killed. This happens with intensity  $\pi_{\Phi'_t}(Y)$ . The death of  $Y$  will remove the pressure  $\pi_{\Phi'_t \cap C}(Y)$  between  $Y$  and  $\Phi'_t \cap C$ , hence the third term.

By standard arguments, we have

$$\mathbb{E} \Pi_{\Phi'_t}(\Phi'_t \cap C) = \beta_{\Phi'_t} |C| \mathbb{E}_{\Phi'_t}^0 \pi_{\Phi'_t}$$

and

$$\mathbb{E} \sum_{X \in \Phi'_t \cap C} \pi_{\Phi'_t}^2(X) = \beta_{\Phi'_t} |C| \mathbb{E}_{\Phi'_t}^0 \pi_{\Phi'_t}^2.$$

Using the mass transport principle (cf Appendix 9.2), we have

$$\mathbb{E} \sum_{Y \in \Phi'_t} \pi_{\Phi'_t \cap C}(Y) \pi_{\Phi'_t}(Y) = \beta_{\Phi'_t} |C| \mathbb{E}_{\Phi'_t}^0 \pi_{\Phi'_t}^2. \quad (32)$$

Hence (31) can be rewritten as indicated in the lemma.  $\square$

**Proposition 2.** *Under the foregoing assumptions, the following differential equations hold for the pressure of regulars on specials:*

$$\begin{aligned} \frac{d}{dt} (\beta_{\mathcal{Z}_t} \mathbb{E}_{\mathcal{Z}_t}^0 [\pi_{\mathcal{R}_t}]) &= \beta_{\mathcal{R}_t} \mathbb{E}_{\mathcal{R}_t}^0 [\pi_{\mathcal{R}_t} \pi_{\mathcal{A}_t}] + \lambda \beta_{\mathcal{Z}_t} a \\ &\quad - \beta_{\mathcal{Z}_t} \mathbb{E}_{\mathcal{Z}_t}^0 [\pi_{\mathcal{R}_t} \pi_{\mathcal{R}_t + \mathcal{Z}_t}] - \beta_{\mathcal{R}_t} \mathbb{E}_{\mathcal{R}_t}^0 [\pi_{\mathcal{Z}_t} \pi_{\mathcal{R}_t + \mathcal{Z}_t + \mathcal{A}_t}], \end{aligned} \quad (33)$$

as well as the symmetrical one (i.e. that for  $\beta_{\mathcal{A}_t} \mathbb{E}_{\mathcal{A}_t}^0 [\pi_{\mathcal{R}_t}]$ ). In addition

$$\frac{d}{dt} (\beta_{\mathcal{S}_t} \mathbb{E}_{\mathcal{S}_t}^0 [\pi_{\mathcal{R}_t}]) = \lambda \beta_{\mathcal{S}_t} a - \beta_{\mathcal{S}_t} \mathbb{E}_{\mathcal{S}_t}^0 [\pi_{\mathcal{R}_t} \pi_{\mathcal{R}_t + \mathcal{S}_t}] - \beta_{\mathcal{R}_t} \mathbb{E}_{\mathcal{R}_t}^0 [(\pi_{\mathcal{S}_t})^2]. \quad (34)$$

For the pressure of specials on specials, we have

$$\frac{d}{dt} (\beta_{\mathcal{Z}_t} \mathbb{E}_{\mathcal{Z}_t}^0 [\pi_{\mathcal{Z}_t}]) = 2\beta_{\mathcal{R}_t} \mathbb{E}_{\mathcal{R}_t}^0 [\pi_{\mathcal{Z}_t} \pi_{\mathcal{A}_t}] - 2\beta_{\mathcal{Z}_t} \mathbb{E}_{\mathcal{Z}_t}^0 [\pi_{\mathcal{Z}_t} \pi_{\mathcal{Z}_t + \mathcal{R}_t}], \quad (35)$$

as well as the symmetrical one (i.e. that for  $\beta_{\mathcal{A}_t} \mathbb{E}_{\mathcal{A}_t}^0 [\pi_{\mathcal{A}_t}]$ ). In addition

$$\frac{d}{dt} (\beta_{\mathcal{S}_t} \mathbb{E}_{\mathcal{S}_t}^0 [\pi_{\mathcal{S}_t}]) = 4\beta_{\mathcal{R}_t} \mathbb{E}_{\mathcal{R}_t}^0 [\pi_{\mathcal{Z}_t} \pi_{\mathcal{A}_t}] - 2\beta_{\mathcal{S}_t} \mathbb{E}_{\mathcal{S}_t}^0 [\pi_{\mathcal{S}_t} \pi_{\mathcal{S}_t + \mathcal{R}_t}]. \quad (36)$$

Finally, for the pressure of regulars on regulars, we have

$$\frac{d}{dt} (\beta_{\mathcal{R}_t} \mathbb{E}_{\mathcal{R}_t}^0 [\pi_{\mathcal{R}_t}]) = \lambda \beta_{\mathcal{R}_t} a - 2\beta_{\mathcal{R}_t} \mathbb{E}_{\mathcal{R}_t}^0 [\pi_{\mathcal{R}_t} \pi_{\mathcal{R}_t + \mathcal{S}_t}]. \quad (37)$$

All terms in the RHSs of these differential equations are uniformy bounded.

*Proof.* The last property is obtained by the same arguments as for densities (see Section 5.1).

For all Borel sets  $C_1, C_2$  and for all point processes  $\mathcal{X}_t, \mathcal{Y}_t$ , consider

$$\Pi_{\mathcal{Y}_t \cap C_2}(\mathcal{X}_t \cap C_1) = \sum_{X \in \mathcal{X}_t \cap C_1} \pi_{\mathcal{Y}_t \cap C_2}(X). \quad (38)$$

We remind the pressure analogy used in the proof of (29):  $\Pi_{\mathcal{Y}_t \cap C_2}(\mathcal{X}_t \cap C_1)$  can be seen as the death pressure exerted by  $\mathcal{Y}_t \cap C_2$  on  $\mathcal{X}_t \cap C_1$ . Note that because of the symmetry of the processes, it is also the pressure exerted by  $\mathcal{X}_t \cap C_1$  on  $\mathcal{Y}_t \cap C_2$ .

We first prove (33). For all sets  $C$ , we have

$$\begin{aligned} \frac{d}{dt} \mathbb{E} \Pi_{\mathcal{R}_t}(\mathcal{Z}_t \cap C) &= \mathbb{E} \sum_{X \in \mathcal{R}_t \cap C} \pi_{\mathcal{R}_t}(X) \pi_{\mathcal{A}_t}(X) \\ &\quad + \lambda a \mathbb{E} |\mathcal{Z}_t \cap C| \\ &\quad - \mathbb{E} \sum_{X \in \mathcal{Z}_t \cap C} \pi_{\mathcal{R}_t}(X) \pi_{\mathcal{R}_t + \mathcal{Z}_t}(X) \\ &\quad - \mathbb{E} \sum_{X \in \mathcal{R}_t} \pi_{\mathcal{Z}_t \cap C}(X) \pi_{\mathcal{R}_t + \mathcal{Z}_t + \mathcal{A}_t}(X). \end{aligned} \quad (39)$$

The reason is the following:  $\Pi_{\mathcal{R}_t}(\mathcal{Z}_t \cap C)$  represents the pressure exerted by  $\mathcal{R}_t$  on  $\mathcal{Z}_t \cap C$ . The reasons for this pressure to change with time are:

- A regular point  $X \in \mathcal{R}_t \cap C$  can be turned into a new zombie due to a regular-antizombie interaction. For each  $X \in \mathcal{R}_t \cap C$ , this happens with intensity  $\pi_{\mathcal{A}_t}(X)$ . The newborn zombie will experience a pressure  $\pi_{\mathcal{R}_t}(X)$  (we recall the convention  $f(0) = 0$ ); hence the first term in (39). This process also removes  $X$  as a regular point, but this effect is considered in the fourth term.
- A new regular point can be born from the Poisson rain process. For each  $Z \in \mathcal{Z}_t \cap C$ , the average pressure increase per time unit due to arrivals is  $\lambda a$ , hence the second term.
- Each zombie  $X \in \mathcal{Z}_t \cap C$  can be killed by a regular point or a zombie. This happens with intensity  $\pi_{\mathcal{R}_t + \mathcal{Z}_t}(X)$ . The death of  $X$  will decrease the total pressure by  $\pi_{\mathcal{R}_t}(X)$ , hence the third term.
- Each regular point  $X \in \mathcal{R}_t$  can be killed by anyone (regular or special). This happens with intensity  $\pi_{\mathcal{R}_t + \mathcal{Z}_t + \mathcal{A}_t}(X)$ . The death of  $X$  will remove the pressure  $\pi_{\mathcal{Z}_t \cap C}(X)$  between  $X$  and  $\mathcal{Z}_t \cap C$ , hence the last term.

Each term in (39) including the one differentiated are uniformly bounded for the same reasons as those used in the proof of Lemma 3 (tightness of the quantities of interest for both initial conditions). From the very definition of Palm probability, we can rewrite the term which is differentiated in (39) as

$$\mathbb{E}\Pi_{\mathcal{R}_t}(\mathcal{Z}_t \cap C) = \beta_{\mathcal{Z}_t}|C|\mathbb{E}_{\mathcal{Z}_t}^0[\pi_{\mathcal{R}_t}]$$

and the first and third terms on the R.H.S. as

$$\begin{aligned}\mathbb{E} \sum_{X \in \mathcal{R}_t \cap C} \pi_{\mathcal{R}_t}(X) \pi_{\mathcal{A}_t}(X) &= \beta_{\mathcal{R}_t}|C|\mathbb{E}_{\mathcal{R}_t}^0[\pi_{\mathcal{R}_t} \pi_{\mathcal{A}_t}] \\ \mathbb{E} \sum_{X \in \mathcal{Z}_t \cap C} \pi_{\mathcal{R}_t}(X) \pi_{\mathcal{R}_t + \mathcal{Z}_t}(X) &= \beta_{\mathcal{Z}_t}|C|\mathbb{E}_{\mathcal{Z}_t}^0[\pi_{\mathcal{R}_t} \pi_{\mathcal{R}_t + \mathcal{Z}_t}],\end{aligned}$$

respectively. In addition, we show in Appendix 9.2 that the following identity holds for the fourth term:

$$\mathbb{E} \sum_{X \in \mathcal{R}_t} \pi_{\mathcal{Z}_t \cap C}(X) \pi_{\mathcal{R}_t + \mathcal{Z}_t + \mathcal{A}_t}(X) = \mathbb{E} \sum_{X \in \mathcal{R}_t \cap C} \pi_{\mathcal{Z}_t}(X) \pi_{\mathcal{R}_t + \mathcal{Z}_t + \mathcal{A}_t}(X). \quad (40)$$

Hence

$$\mathbb{E} \sum_{X \in \mathcal{R}_t} \pi_{\mathcal{Z}_t \cap C}(X) \pi_{\mathcal{R}_t + \mathcal{Z}_t + \mathcal{A}_t}(X) = \beta_{\mathcal{R}_t}|C|\mathbb{E}_{\mathcal{R}_t}^0[\pi_{\mathcal{Z}_t} \pi_{\mathcal{R}_t + \mathcal{Z}_t + \mathcal{A}_t}]. \quad (41)$$

We get (33) when dividing (39) by  $|C|$ . The other equations are obtained in the same way.  $\square$

Here are a few observations on these equations. Consider e.g. (34). The only positive term in the RHS of this equation is  $\lambda\beta_{\mathcal{S}_t}a$ . In particular, the positive term  $\beta_{\mathcal{R}_t}\mathbb{E}_{\mathcal{R}_t}^0[\pi_{\mathcal{R}_t}\pi_{\mathcal{S}_t}]$  (contamination of an  $R$  by an  $S$ ) is nullified. The reason is obvious if one considers the death of a typical  $R \in \mathcal{R}_t$ .  $R$  undergoes a pressure  $\pi_{\mathcal{R}_t}(R)$  from  $\mathcal{R}_t$  and  $\pi_{\mathcal{S}_t}(R)$  from  $\mathcal{S}_t$ . If the killing comes from the pressure of  $\mathcal{S}_t$ , a new special is created in  $R$  and the pressure from  $\mathcal{R}_t$  will be *added* to the pressure between  $\mathcal{S}_t$  and  $\mathcal{R}_t$ <sup>8</sup>. Conversely, if the killing comes from the pressure of  $\mathcal{R}_t$ ,  $R$  is removed and its pressure from  $\mathcal{S}_t$  is *subtracted*.

It is not difficult to check the following consistency property: when adding twice (33), (35) and (37), we get back (29) as expected.

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<sup>8</sup>Meanwhile, the pressure  $\pi_{\mathcal{S}_t}(R)$  is also removed, hence the  $-(\pi_{\mathcal{S}_t}(R))^2$  term.



**Proposition 3.** *Under the foregoing assumptions, both  $\mathbb{E}_{\mathcal{S}_t}^0 \pi_{\mathcal{R}_t}$  and  $\mathbb{E}_{\mathcal{S}_t}^0 \pi_{\mathcal{S}_t}$  are uniformly bounded with respect to  $t$ .*

*Proof.* By adding twice (34) and (36), we get

$$\begin{aligned} \frac{d}{dt} (\beta_{\mathcal{S}_t} \mathbb{E}_{\mathcal{S}_t}^0 [2\pi_{\mathcal{R}_t} + \pi_{\mathcal{S}_t}]) &= 2\lambda\beta_{\mathcal{S}_t}a - 2\beta_{\mathcal{S}_t} \mathbb{E}_{\mathcal{S}_t}^0 [(\pi_{\mathcal{R}_t} + \pi_{\mathcal{S}_t})^2] \\ &\quad - 2\beta_{\mathcal{R}_t} \mathbb{E}_{\mathcal{R}_t}^0 [(\pi_{\mathcal{A}_t} + \pi_{\mathcal{Z}_t})^2] + 4\beta_{\mathcal{R}_t} \mathbb{E}_{\mathcal{R}_t}^0 [\pi_{\mathcal{Z}_t} \pi_{\mathcal{A}_t}] \\ &\leq 2\lambda\beta_{\mathcal{S}_t}a - 2\beta_{\mathcal{S}_t} \mathbb{E}_{\mathcal{S}_t}^0 [(\pi_{\mathcal{R}_t} + \pi_{\mathcal{S}_t})^2]. \end{aligned}$$

By making use of (27) in the last equation, we get

$$\begin{aligned} \frac{d}{dt} \mathbb{E}_{\mathcal{S}_t}^0 [2\pi_{\mathcal{R}_t} + \pi_{\mathcal{S}_t}] &\leq 2\lambda a + \mathbb{E}_{\mathcal{S}_t}^0 [\pi_{\mathcal{S}_t}] \mathbb{E}_{\mathcal{S}_t}^0 [2\pi_{\mathcal{R}_t} + \pi_{\mathcal{S}_t}] - 2\mathbb{E}_{\mathcal{S}_t}^0 [(\pi_{\mathcal{R}_t} + \pi_{\mathcal{S}_t})^2] \\ &\leq 2\lambda a + (\mathbb{E}_{\mathcal{S}_t}^0 [\pi_{\mathcal{S}_t} + \pi_{\mathcal{R}_t}])^2 - 2\mathbb{E}_{\mathcal{S}_t}^0 [(\pi_{\mathcal{R}_t} + \pi_{\mathcal{S}_t})^2] \\ &\leq 2\lambda a - (\mathbb{E}_{\mathcal{S}_t}^0 [\pi_{\mathcal{R}_t} + \pi_{\mathcal{S}_t}])^2 \end{aligned}$$

Thus  $\mathbb{E}_{\mathcal{S}_t}^0 [2\pi_{\mathcal{R}_t} + \pi_{\mathcal{S}_t}]$  is decreasing whenever  $\mathbb{E}_{\mathcal{S}_t}^0 [\pi_{\mathcal{R}_t} + \pi_{\mathcal{S}_t}] > \sqrt{2\lambda a}$ . As  $\mathbb{E}_{\mathcal{S}_t}^0 [2\pi_{\mathcal{R}_t} + \pi_{\mathcal{S}_t}] > 2\sqrt{2\lambda a}$  implies  $\mathbb{E}_{\mathcal{S}_t}^0 [\pi_{\mathcal{R}_t} + \pi_{\mathcal{S}_t}] > \sqrt{2\lambda a}$ , we get

$$\limsup_{t \rightarrow \infty} \mathbb{E}_{\mathcal{S}_t}^0 [2\pi_{\mathcal{R}_t} + \pi_{\mathcal{S}_t}] \leq 2\sqrt{2\lambda a}.$$

□

## 7 Construction of the Stationary Regime

The most important result of Section 4 for the construction of the stationary regime is Theorem 3, which shows that the perturbation induced by any point (in a finite intensity augmentation point process) a.s. vanishes with time. If we were on a finite domain, there would be a finite number of additional points, each with an influence that vanishes in finite time and the processes with and without augmentation would hence coincide after a finite time. This would provide a natural way of proving the existence and the uniqueness of the stationary regimes through a coupling from the past construction. This is the line of thought that we follow. The main technical difficulty consists in proving that the influence of the additional points vanishes fast enough.

In this section, we assume that the initial condition is a thinned homogeneous Poisson point process of finite intensity.

## 7.1 Exponential Decay of the Density of Special Points

In this subsection, we suppose that Assumptions 0–3 hold. We will use the following notation for  $\sigma$ -algebras:

$$\begin{aligned}\mathcal{D}_t &= \sigma(\mathcal{Z}_0 \cup \Psi_{(0,t]}) \\ \mathcal{T}_t &= \sigma(T_{pq} : p, q \in \mathcal{Z}_0 \cup \Psi_{(0,t]}) \\ \mathcal{I}_t &= \sigma(I_{pq} : p, q \in \mathcal{Z}_0 \cup \Psi_{(0,t]}) \\ \mathcal{G}_t &= \mathcal{D}_t \vee \mathcal{T}_t \vee \mathcal{I}_t.\end{aligned}$$

Note that  $\tilde{\Phi}_t$ , introduced in Section 4.3, is  $\mathcal{G}_t$ -measurable.

For any special node  $z \in \tilde{\Phi}_t$ , denote by  $M_{z,t,s}$  the number of points that are offsprings of  $z$  and that are still alive at time  $s \geq t$  (here when  $z$  kills an ordinary point, the latter becomes a first generation offspring of  $z$ ; when this node kills another ordinary point, the latter is seen as a second generation offspring of  $z$ , etc). The following lemma is a direct corollary of what was already established in the proofs of Theorems 2 and 3:

**Lemma 8.** *Under the assumptions of Theorem 2, for all special nodes  $z \in \tilde{\Phi}_t$  and all  $s \geq t$ ,*

$$\mathbb{E}[M_{z,t,s} | \mathcal{G}_t \vee \mathcal{D}_s \vee \mathcal{T}_s] \leq 1.$$

For any  $t > 0$ , any special point  $z \in \tilde{\Phi}_t$  and any fixed positive numbers  $r$  and  $\epsilon$ , consider the event

$$A_1 = \{\Psi_{(t,t+\epsilon)}(B(z, r)) = 2\}.$$

On  $A_1$ , denote by  $(a, t_a), (b, t_b)$  the random locations of the two points of  $\Psi_{(t,t+\epsilon)} \cap B(z, r)$  and consider the events

$$\begin{aligned}A_2 &= \{t_a \vee t_b < T_{za} < T_{zb} < T_{ab} < t + \epsilon\}, \\ A_3 &= \left\{ \forall p \in \{z, a, b\}, \forall q \in \tilde{\Phi}_t \cup e_1(\Psi_{(t,t+\epsilon)}) \cap B(z, r)^c : T_{pq} > t + \epsilon \right\},\end{aligned}$$

where  $e_1(\Psi)$  denotes the projection  $\mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^d$ . Let  $G = A_1 \cap A_2 \cap A_3$ . Notice that  $G$  is in  $\mathcal{G}_t \vee \mathcal{D}_{t+\epsilon} \vee \mathcal{T}_{t+\epsilon}$ .

**Lemma 9.** *Under Assumptions 0–3, for all  $t, \epsilon > 0$ , and for all special points  $z \in \tilde{\Phi}_t$ ,*

$$\mathbb{E}[M_{z,t,t+\epsilon} | \mathcal{G}_t] \leq 1 - \frac{1}{4}\mathbb{P}[G | \mathcal{G}_t].$$

*Proof.* On  $G$ , the value of  $M_{z,t,t+\epsilon}$  depends only on the random variables  $I_{za}$ ,  $I_{zb}$  and  $I_{ab}$ . If  $I_{za} = -1$ ,  $z$  is killed without producing any offspring after  $t$ . If  $I_{za} = 1$  and  $I_{zb} = -1$ , first  $a$  becomes special and then  $b$  kills  $z$ . If then  $I_{ab} = 1$ , both  $a$  and  $b$  become special, whereas in the opposite case  $z$ 's family dies out. Finally, if  $I_{za} = I_{zb} = 1$ , both  $a$  and  $b$  become special of same kind and, at time  $T_{ab}$ , one of them kills the other. Thus the expected value of  $M_{z,t,t+\epsilon}$  on  $G$  is  $(4 \cdot 0 + 1 \cdot 2 + 1 \cdots 0 + 2 \cdot 2)/8 = \frac{3}{4}$ . Hence

$$\mathbb{E} [1_{\{G\}} M_{z,t,t+\epsilon} \mid \mathcal{G}_t] = \frac{3}{4} \mathbb{P}[G \mid \mathcal{G}_t].$$

Similarly,

$$\begin{aligned} \mathbb{E} [1_{\{G^c\}} M_{z,t,t+\epsilon} \mid \mathcal{G}_t] &= \mathbb{E} [1_{\{G^c\}} \mathbb{E} [M_{z,t,t+\epsilon} \mid \mathcal{D}_{t+\epsilon} \vee \mathcal{T}_{t+\epsilon} \vee \mathcal{G}_t] \mid \mathcal{G}_t] \\ &\leq \mathbb{E} [1_{\{G^c\}} \mid \mathcal{G}_t], \end{aligned}$$

where we used Lemma 8. Hence

$$\begin{aligned} \mathbb{E} [M_{z,t,t+\epsilon} \mid \mathcal{G}_t] &= \mathbb{E} [1_{\{G\}} M_{z,t,t+\epsilon} \mid \mathcal{G}_t] + \mathbb{E} [1_{\{G^c\}} M_{z,t,t+\epsilon} \mid \mathcal{G}_t] \\ &= \frac{3}{4} \mathbb{P}[G \mid \mathcal{G}_t] + \mathbb{E} [1_{\{G^c\}} M_{z,t,t+\epsilon} \mid \mathcal{G}_t] \\ &\leq \frac{3}{4} \mathbb{P}[G \mid \mathcal{G}_t] + 1 - \mathbb{P}[G \mid \mathcal{G}_t]. \end{aligned}$$

□

From classical properties of Poisson point processes, reminding that  $\nu_d$  denotes the volume of a unit ball in  $\mathbb{R}^d$ , we get

$$\begin{aligned} \mathbb{P}[G \mid \mathcal{G}_t] &= \mathbb{P}[A_1] \frac{1}{(\nu_d r^d \epsilon)^2} \int_{B(z,r)} \int_{B(z,r)} \int_{[t,t+\epsilon]} \int_{[t,t+\epsilon]} \\ &\quad \mathbb{P}[u \vee v < T_{zx} < T_{zy} < T_{xy} < t + \epsilon] \times \\ &\quad \mathbb{P} \left[ \forall p \in \{z, x, y\}, \forall q \in \tilde{\Phi}_t \cup e_1(\Psi_{(t,t+\epsilon)}) \cap B(z,r)^c : T_{pq} > t + \epsilon \mid \mathcal{G}_t \right] \\ &\quad dxdydu dv. \end{aligned}$$

Notice that  $\mathbb{P}[A_1]$  is a constant that does not depend on  $t$ . Similarly, if  $r$  is such that  $f(2r) > 0$ , then  $\mathbb{P}[u \vee v < T_{zx} < T_{zy} < T_{xy} < t + \epsilon]$  is bounded from below by a constant that does not depend on  $t, u, v, x, y$ . From this and the independence properties of the Poisson rain after  $t$  and  $\mathcal{G}_t$ , we get that for all  $r$  as above, there exists a constant  $0 < F(r, \epsilon) < 1$  such that

$$\mathbb{P}[G \mid \mathcal{F}_t] \geq F(r, \epsilon) \frac{1}{(\nu_d r^d)^2} \int_{B(z,r)} \int_{B(z,r)}$$

$$\mathbb{P}\left[\forall p \in \{z, x, y\}, \forall q \in \tilde{\Phi}_t : T'_{pq} > t + \epsilon \mid \mathcal{G}_t\right] dx dy, \quad (42)$$

where the random variables  $T'_{pq}$  are mutually independent and  $T'_{pq}$  is  $t$  plus an exponential of parameter  $2f(\|p - q\|)$ .

**Theorem 4.** *Under Assumptions 0–3, there exists an  $\alpha > 0$  such that*

$$\beta_{\mathcal{S}_t} \leq e^{-\alpha t} \quad (43)$$

for  $t$  large enough.

*Proof.* From Proposition 3, there exists a  $J < \infty$  such that

$$\mathbb{E}_{\mathcal{S}_t}^0[\pi_{\mathcal{S}_t}(0) + \pi_{\mathcal{R}_t}(0)] < J, \quad (44)$$

uniformly in  $t$ . Since the point process  $\mathcal{S}_t$  is spatially stationary, we can use the Campbell–Mecke formula to prove that for all  $\epsilon > 0$ ,

$$\beta_{\mathcal{S}_{t+\epsilon}} = \beta_{\mathcal{S}_t} \mathbb{E}_{\mathcal{S}_t}^0[M_{0,t,t+\epsilon}]$$

with  $M_{0,t,s}$  the number of special points offspring of the origin living at time  $s$ , under  $\mathbb{P}_{\mathcal{S}_t}^0$ . From Lemma 9 and (42), we get

$$\beta_{\mathcal{S}_{t+\epsilon}} \leq \beta_{\mathcal{S}_t} - \beta_{\mathcal{S}_t} \frac{1}{4} F(r, \epsilon) \xi_t$$

with

$$\xi_t = \frac{1}{(\nu_d r^d)^2} \int_{B(0,r)} \int_{B(0,r)} \mathbb{P}_{\mathcal{S}_t}^0 \left[ \forall p \in \{0, x, y\}, \forall q \in \tilde{\Phi}_t^0 : T'_{pq} > t + \epsilon \right] dx dy.$$

Let us show that when (44) holds, there exists an  $r$ , a  $\epsilon$  and constant  $0 < C(r, \epsilon) \leq 1$  that does not depend on  $t$  and such that

$$\xi_t > C(r, \epsilon). \quad (45)$$

We first explain the idea of the proof of (45) by ignoring the conditions on  $x$  and  $y$ .

$$\begin{aligned} \mathbb{P}_{\mathcal{S}_t}^0[T'_{0q} > t + \epsilon, \forall q \neq 0 \in \tilde{\Phi}_t^0] &= \mathbb{E}_{\mathcal{S}_t}^0 \left[ \mathbb{P}_{\mathcal{S}_t}^0[T'_{0q} > \epsilon, \forall q \neq 0 \in \tilde{\Phi}_t^0 \mid \tilde{\Phi}_t] \right] \\ &= \mathbb{E}_{\mathcal{S}_t}^0 \left[ \prod_{q \neq 0 \in \tilde{\Phi}_t^0} e^{-2\epsilon f(\|q\|)} \right] \end{aligned}$$

$$\begin{aligned}
&\geq 1 - \epsilon \mathbb{E}_{\mathcal{S}_t}^0 \left[ \sum_{q \neq 0 \in \tilde{\Phi}_t^0} 2f(\|q\|) \right] \\
&= 1 - 2\epsilon (\mathbb{E}_{\mathcal{S}_t}^0 \pi_{\mathcal{S}_t}(0) + \mathbb{E}_{\mathcal{S}_t}^0 \pi_{\mathcal{R}_t}(0)) \\
&\geq 1 - 2\epsilon J.
\end{aligned}$$

We now consider  $x$  and  $y$  in addition to 0. By the same arguments, we have

$$\begin{aligned}
&\mathbb{P}_{\mathcal{S}_t}^0 \left[ \forall p \in \{0, x, y\}, \forall q \in \tilde{\Phi}_t^0 : T'_{pq} > t + \epsilon \right] \\
&= \mathbb{E}_{\mathcal{S}_t}^0 \left[ \prod_{q \neq 0 \in \tilde{\Phi}_t^0} e^{-2\epsilon(f(\|q\|) + f(\|q-x\|) + f(\|q-y\|))} \right] \\
&\geq 1 - 2\epsilon (\mathbb{E}_{\mathcal{S}_t}^0 \pi_{\mathcal{S}_t}(0) + \mathbb{E}_{\mathcal{S}_t}^0 \pi_{\mathcal{S}_t}(x) + \mathbb{E}_{\mathcal{S}_t}^0 \pi_{\mathcal{S}_t}(y) \\
&\quad + \mathbb{E}_{\mathcal{S}_t}^0 \pi_{\mathcal{R}_t}(0) + \mathbb{E}_{\mathcal{S}_t}^0 \pi_{\mathcal{R}_t}(x) + \mathbb{E}_{\mathcal{S}_t}^0 \pi_{\mathcal{R}_t}(y)). \quad (46)
\end{aligned}$$

Let us now show that, under the foregoing assumptions, if  $\mathbb{E}_{\mathcal{S}_t}^0 \pi_{\mathcal{S}_t}(0)$  is uniformly bounded, then so are  $\mathbb{E}_{\mathcal{S}_t}^0 \pi_{\mathcal{S}_t}(x)$  and  $\mathbb{E}_{\mathcal{S}_t}^0 \pi_{\mathcal{S}_t}(y)$ . The initial condition satisfies the assumptions of Section 3.2 and arrivals in  $(0, t)$  form a marked Poisson point process on  $\mathbb{R}^d$ . Both are motion-invariant. Motion invariance is preserved by the dynamics. Hence we have

$$\mathbb{E}_{\mathcal{S}_t}^0 \pi_{\mathcal{S}_t}(0) = \nu_d d \frac{1}{\beta_{\mathcal{S}_t}} \int_{r>0} f(r) \rho_{\mathcal{S}_t}^{[2]}(r) r^{d-1} dr,$$

where  $\rho_{\mathcal{S}_t}^{[2]}(r)$  is the radial component of the (motion invariant) density of the reduced second moment measure of  $\mathcal{S}_t$  (see Section 8.1 for definitions). The fact that the last function is uniformly bounded implies that  $\mathbb{E}_{\mathcal{S}_t}^0 \mathcal{S}_t(B(0, b))$  is uniformly bounded for all  $b$  such that  $f(b) > 0$ . It also implies that for all  $H > 1$ ,

$$\nu_d d \frac{1}{\beta_{\mathcal{S}_t}} \int_{r>0} f\left(\frac{r}{H}\right) \rho_{\mathcal{S}_t}^{[2]}\left(\frac{r}{H}\right) r^{d-1} dr$$

is uniformly bounded. For all  $x$  with  $\|x\| < b$ , with  $b$  such that  $f(b) > 0$ , from monotonicity and boundedness, we have

$$\mathbb{E}_{\mathcal{S}_t}^0 \pi_{\mathcal{S}_t}(x) \leq K \mathbb{E}_{\mathcal{S}_t}^0 [\mathcal{S}_t(B(0, b))] + \nu_d d \frac{1}{\beta_{\mathcal{S}_t}} \int_{r \geq \rho} f\left(\frac{r}{H}\right) \rho_{\mathcal{S}_t}^{[2]}\left(\frac{r}{H}\right) r^{d-1} dr,$$

with  $H = \frac{b}{b-\|x\|}$ , which shows that  $\mathbb{E}_{\mathcal{S}_t}^0 \pi_{\mathcal{S}_t}(x)$  is uniformly bounded. By similar arguments, since  $\mathbb{E}_{\mathcal{S}_t}^0 \pi_{\mathcal{R}_t}(0)$  is uniformly bounded, then so are  $\mathbb{E}_{\mathcal{S}_t}^0 \pi_{\mathcal{R}_t}(x)$  and  $\mathbb{E}_{\mathcal{S}_t}^0 \pi_{\mathcal{R}_t}(y)$ .

Thus, thanks to the uniform boundedness of all 6 terms showing up in (46), one can choose a  $\epsilon$  and a  $r$  small enough in (46) for (45) to hold.

Hence, for all  $t$ ,

$$\beta_{\mathcal{S}_{t+\epsilon}} \leq \beta_{\mathcal{S}_t} - \beta_{\mathcal{S}_t} \gamma,$$

with  $0 < \gamma = \gamma(r, \epsilon) = \frac{1}{4}F(r, \epsilon)C(r, \epsilon) < 1$ .

Since the function  $t \rightarrow \beta_{\mathcal{S}_t}$  is monotone non-increasing, for  $\epsilon > 0$  as defined above,

$$\beta_{\mathcal{S}_t} \leq \beta_{\mathcal{S}_{\epsilon \lfloor \frac{t}{\epsilon} \rfloor}} \leq \beta_{\mathcal{S}_0} (1 - \gamma)^{\lfloor \frac{t}{\epsilon} \rfloor},$$

with the last inequality following from the above bound. Since  $\lfloor \frac{t}{\epsilon} \rfloor \geq -1 + \frac{t}{\epsilon}$ , it follows that

$$\beta_{\mathcal{S}_t} \leq \frac{\beta_{\mathcal{S}_0}}{1 - \gamma} (1 - \gamma)^{\frac{t}{\epsilon}},$$

for all  $t$ , which concludes the proof.  $\square$

**Theorem 5.** *Consider two executions of  $\text{Sheriff}^Z$ : that with an empty initial condition and that with a stationary and ergodic initial point process  $\mathcal{Z}_0$  which satisfies the conditions of Subsection 3.2. Under Assumptions 0–3, for all compacts  $C$  of  $\mathbb{R}^d$ , there exists a random time  $\tau(C)$  with finite expectation such that for all  $t \geq \tau(C)$ , these two executions coincide in  $C$ .*

*Proof.* Denote by  $N_t = \mathcal{S}_t(C)$  the number of special points living in  $C$  at time  $t$ . It suffices to show that the random time

$$\tau(C) = \sup \{t \geq 0 : N_t > 0\}$$

has finite expectation. Note first that Theorem 4 already yields

$$\mathbb{E} \int_0^\infty N_t dt = \int_0^\infty \mathbb{E} N_t dt = |C| \int_0^\infty \beta_{\mathcal{S}_t} dt < \infty. \quad (47)$$

Write

$$N_t = N_0 + N_t^+ - N_t^-,$$

where  $N^+$  and  $N^-$  are the counting processes of births and deaths of special points in  $C$ . Since the stochastic intensity of  $N^+$ , say  $\lambda_t^{N^+}$ , is

$$\lambda_t^{N^+} = \sum_{x \in \mathcal{R}_t \cap C} \pi_{\mathcal{S}_t}(x),$$

we have, using the mass transport principle,

$$\mathbb{E}\lambda_t^{N^+} = \beta_{\mathcal{R}_t}|C|\mathbb{E}_{\mathcal{R}_t}^0\pi_{\mathcal{S}_t} = \beta_{\mathcal{S}_t}|C|\mathbb{E}_{\mathcal{S}_t}^0\pi_{\mathcal{R}_t}. \quad (48)$$

Denote the sorted birth and death times of special points in  $C$  by  $(T_n^{N^+})_{n \geq 1}$  and  $(T_n^{N^-})_{n \geq 1}$ , respectively. By the definition of stochastic intensity, (48), Proposition 3 and Theorem 4,

$$\begin{aligned} \mathbb{E} \sum_{n=1}^{\infty} T_n^{N^+} &= \mathbb{E} \int_0^{\infty} t \, dN_t^+ \\ &= \mathbb{E} \int_0^{\infty} t \lambda_t^{N^+} \, dt \\ &= \int_0^{\infty} t \mathbb{E} \lambda_t^{N^+} \, dt \\ &\leq |C| (\sup_t \mathbb{E}_{\mathcal{S}_t}^0 \pi_{\mathcal{R}_t}) \int_0^{\infty} t \beta_{\mathcal{S}_t} \, dt < \infty. \end{aligned} \quad (49)$$

We can write

$$\int_0^{\infty} N_t \, dt = \sum_{n=1}^{\infty} T_n^{N^-} - \sum_{n=1}^{\infty} T_n^{N^+},$$

since the last sum is finite by (49). Now the claim follows by noting that

$$\mathbb{E}\tau(C) \leq \mathbb{E} \sum_{n=1}^{\infty} T_n^{N^-} = \mathbb{E} \int_0^{\infty} N_t \, dt + \mathbb{E} \sum_{n=1}^{\infty} T_n^{N^+} < \infty.$$

□

## 7.2 Coupling from the Past

Throughout this section Assumptions 0–3 are supposed to hold and  $\mathcal{Z}_0$  is a translation invariant initial condition which satisfies the properties listed in Subsection 3.2.

For each location  $y \in \mathbb{R}^d$ , let  $V_y$  denote the time it takes for the point process generated by Sheriff acting on  $\Psi_{(0,\infty)}$  with the empty initial condition, and that generated by Sheriff acting on  $\Psi_{(0,\infty)}$  with the initial condition  $\mathcal{Z}_0$  to couple (i.e. to be identical forever) in the unit ball centered at  $y$ . Under the foregoing assumptions, the random field  $V_y$  is translation invariant. From Theorem 5, for all  $y$ ,  $\mathbb{E}V_y = \mathbb{E}V_0 < \infty$ .

Now, in Theorem 5, choose

$$\mathcal{Z}_0 := \text{the nodes of } \Psi_{(-1,0]} \text{ alive after running Sheriff on them,}$$

as augmentation of the initial condition of Sheriff acting on  $\Psi_{(0,\infty)}$ . Let  $V_y^{(0)} = V_y$  denote the associated coupling time field. Consider also the set of nodes of  $\Psi_{(-2,-1]}$  still alive at time -1 as augmentation of the initial condition of Sheriff acting on  $\Psi_{(-1,\infty)}$ , and denote by  $V_y^{(1)}$  the associated coupling time field. The random fields  $V_y^{(0)}$  and  $V_y^{(1)} + 1$  are stochastically equivalent: denoting by  $\theta_t$  the measure preserving time shift

$$\theta_t \Psi(C \times H) = \Psi(C \times (t + H)),$$

for all Borel sets  $C$  of  $\mathbb{R}^d$  and all Borel sets  $H$  of  $\mathbb{R}$ , we get that

$$V_y^{(0)} \circ \theta_{-1} = V_y^{(1)} + 1, \quad \forall y.$$

Continuing like this, we obtain a sequence  $(V_y^{(n)} + n)_{n \in \mathbb{N}}$  of identically distributed, and spatially stationary fields.

**Lemma 10.** *Under Assumptions 0-3, for all  $y \in \mathbb{R}^d$ ,  $V_y^{(n)} \rightarrow -\infty$  a.s. when  $n \rightarrow \infty$ .*

*Proof.* Rewrite  $V_y^{(n)}$  as  $V_y^{(n)} + n - n$ . Since the sequence  $V_y^{(n)} + n$  is stationary and ergodic with finite mean,  $\frac{V_y^{(n)} + n}{n} \rightarrow 0$  when  $n \rightarrow \infty$ . This implies the announced result.  $\square$

The following theorem builds a *time stationary* family of point processes, compatible with the birth and death dynamics.

**Theorem 6.** *For  $t \in \mathbb{R}$ , let  $\Phi_{(-n,t)}^\emptyset(t)$  denote the point process of nodes that Sheriff builds alive at time  $t$ , when starting the dynamics at time  $-n$  and with an empty initial condition. For all  $t$ , the a.s. limit*

$$\Upsilon_t = \lim_{n \rightarrow \infty} \Phi_{(-n,t)}^\emptyset(t) \tag{50}$$

*exists and forms a time-stationary family of translation invariant point processes on  $\mathbb{R}^d$ .*

*Proof.* From Lemma 10, for all compacts  $C$  of  $\mathbb{R}^d$ , for all  $t$ , when  $n$  tends to  $\infty$ ,  $\Phi_{(-n,t)}^\emptyset(t)$ , couples with a finite random variable  $\Upsilon_t$  for  $n$  larger than a finite random threshold, denoted by  $\tau_t(C)$ , so that the limit (50) a.s. exists indeed. The property that  $(\Upsilon_t)_{t \in \mathbb{R}}$  is a time-stationary family of point processes follows from the fact that for all  $t$ ,  $\Upsilon_t = \Upsilon_0 \circ \theta_t$ .  $\square$



The following theorem completes the analysis of the stationary regimes in terms of convergence in distribution to  $\Upsilon_0$ . We recall that a sequence of point processes  $\phi_n$  converges in distribution to the point process  $\phi$  if and only if for any function  $h : \mathbb{R}^d \rightarrow \mathbb{R}^+$ , which has bounded support and is continuous,  $\int h d\phi_n$  converges in distribution to  $\int h d\phi$  in  $\mathbb{R}^+$  [6].

**Theorem 7.** *Let  $\Phi_{[0,n)}^{\mathcal{Z}_0}(n)$  be the point process of nodes living at time  $n$  constructed by Sheriff when starting at time 0 with some initial condition  $\mathcal{Z}_0$ . Under Assumptions 0–3, for all initial conditions  $\mathcal{Z}_0$  satisfying the assumptions of Subsection 3.2,  $\Phi_{[0,n)}^{\mathcal{Z}_0}(n)$  converges in distribution to  $\Upsilon_0$ .*

*Proof.* Let us first show that  $\Phi_{[0,n)}^\emptyset(n)$  converges in distribution to  $\Upsilon_0$  when  $n$  tends to  $\infty$ . From Theorem 6, for all functions  $h$  with bounded support  $C$  (in particular continuous),

$$\sup_{A \in \mathcal{B}(\mathbb{R})} \left| \mathbb{P}\left[\int h d\Phi_{[-n,0)}^\emptyset(0) \in A\right] - \mathbb{P}\left[\int h d\Upsilon_0 \in A\right] \right| \leq \mathbb{P}[n < \tau_0(C)] \rightarrow 0 \quad (51)$$

when  $n \rightarrow \infty$ . Since  $\Phi_{[0,n)}^\emptyset(n) = \Phi_{[-n,0)}^\emptyset(0) \circ \theta_n$ , we can replace  $\Phi_{[-n,0)}^\emptyset(0)$  by  $\Phi_{[0,n)}^\emptyset(n)$  in (51), which proves the convergence in distribution of  $\Phi_{[0,n)}^\emptyset(n)$  to  $\Upsilon_0$ . By arguments similar to those of Theorem 6, one gets from Theorem 5 that for initial conditions  $\mathcal{Z}_0$  as above, and for all  $C$  compact,  $\Phi_{[-n,0)}^\emptyset$  and  $\Phi_{[-n,0)}^{\mathcal{Z}_0}$  couple on  $C$  for  $n$  larger than a finite threshold. This in turn implies that  $\Phi_{[-n,0)}^{\mathcal{Z}_0}$  and  $\Upsilon_0$  couple on  $C$  for  $n$  larger than a finite threshold. By arguments similar to those above, this finally implies that  $\Phi_{[0,n)}^{\mathcal{Z}_0}$  converges in distribution to  $\Upsilon_0$ .  $\square$

## 8 Balance Equations for Moment Measures

The aim of this section is to establish a hierarchy of integral relations between the higher order factorial moment measures of the steady state SBD process  $\Upsilon = \Upsilon_0$  on  $\mathbb{R}^d$  constructed in the previous sections. We will denote the factorial moment density of order  $k$  by  $\rho^{[k]}(x_1, \dots, x_k)$ . Notice that  $\Upsilon$  is motion invariant (stationary and isotropic). Hence  $\rho^{[1]}(x) = \beta$  (the intensity of  $\Upsilon$ ),

$$\rho^{[2]}(x, y) = \rho_{st}^{[2]}(x - y) = \rho_{mi}^{[2]}(\|x - y\|)$$

and for all  $k \geq 2$ ,

$$\rho^{[k]}(x_1, \dots, x_k) = \rho_{st}^{[k]}(x_2 - x_1, \dots, x_k - x_1).$$

We first establish these balance equations. For  $k \geq 2$ , the  $k$ -balance equation relates the  $k$ -th to the  $k - 1$ -st and the  $k + 1$ -st factorial moment density. For  $k = 1$ , it relates the first and the second densities.

Then we show how to use these equations to get bounds and approximations.

## 8.1 Balance Equations

For all  $f$  positive, if  $\Upsilon = \sum_n \delta_{X_n}$ , we have

$$\mathbb{E}_{\Upsilon}^0 \left[ \sum_{n \neq 0} f(\|X_n\|) \right] = \frac{1}{\beta} \int_{\mathbb{R}^d} f(\|x\|) \rho_{mi}^{[2]}(\|x\|) dx = \frac{d\nu_d}{\beta} \int_{\mathbb{R}^+} f(r) \rho_{mi}^{[2]}(r) r^{d-1} dr,$$

with  $\nu_d$  the volume of the unit ball in  $\mathbb{R}^d$ .

In steady state the mean number of deaths in a Borel set  $C$  in the time interval  $[0, \epsilon]$  is

$$\beta|C|\epsilon \mathbb{E}_{\Upsilon}^0 \left[ \sum_{n \neq 0} f(\|X_n\|) \right] + o(\epsilon),$$

and it should be equal to the mean number of births in this set and time interval, which is  $\lambda|C|\epsilon$ . We get from this the following relation:

$$\int_{\mathbb{R}^d} \rho_{mi}^{[2]}(\|x\|) f(\|x\|) dx = \int_{\mathbb{R}^d} \rho_{st}^{[2]}(x) f(\|x\|) dx = \lambda \quad (52)$$

which is our first balance relation which links the first and the second order factorial moment densities.

Let  $C_1, C_2$  be two Borel sets. Let us look as above at the mean increase (due to births) and the mean decrease (due to deaths) of the following quantity:

$$\mathbb{E} \sum_{x_1 \neq x_2 \in \Upsilon} 1_{C_1}(x_1) 1_{C_2}(x_2) = \int_{C_1} \int_{C_2} \rho^{[2]}(x_1, x_2) dx_1 dx_2.$$

The mean increase in an interval of time of length  $\epsilon$  is easily seen to be

$$\lambda\epsilon|C_1|\beta|C_2| + \lambda\epsilon|C_2|\beta|C_1| + o(\epsilon).$$

The mean decrease in the same interval is

$$\mathbb{E} \sum_{x_1 \neq x_2 \in \Upsilon} 1_{C_1}(x_1) 1_{C_2}(x_2) \epsilon \left( \sum_{z \in \Upsilon, z \neq x_1} f(\|z - x_1\|) + \sum_{z \in \Upsilon, z \neq x_2} f(\|z - x_2\|) \right)$$

$$\begin{aligned}
& + o(\epsilon) \\
& = 2\epsilon \mathbb{E} \sum_{x_1 \neq x_2 \in \Upsilon} 1_{C_1}(x_1) 1_{C_2}(x_2) f(\|x_1 - x_2\|) \\
& + \epsilon \mathbb{E} \sum_{x_1, x_2, z \in \Upsilon, \text{ different}} 1_{C_1}(x_1) 1_{C_2}(x_2) (f(\|z - x_1\|) + f(\|z - x_2\|)) + o(\epsilon) \\
& = 2\epsilon \int_{C_1} \int_{C_2} f(\|x_1 - x_2\|) \rho^{[2]}(x_1, x_2) dx_1 dx_2 \\
& + \epsilon \int_{C_1} \int_{C_2} \int_{\mathbb{R}^d} f(\|z - x_1\|) + f(\|z - x_2\|) \rho^{[3]}(x_1, x_2, z) dx_1 dx_2 dz + o(\epsilon).
\end{aligned}$$

Hence

$$\begin{aligned}
& 2\rho^{[2]}(x_1, x_2) f(\|x_1 - x_2\|) + \int_{\mathbb{R}^d} \rho^{[3]}(x_1, x_2, z) (f(\|x_1 - z\|) + f(\|x_2 - z\|)) dz \\
& = 2\beta\lambda,
\end{aligned} \tag{53}$$

that is, for all  $x \in \mathbb{R}^d$ :

$$2\rho_{mi}^{[2]}(\|x\|) f(\|x\|) + \int_{\mathbb{R}^d} \rho_{st}^{[3]}(x, y) (f(\|y\|) + f(\|y - x\|)) dy = 2\beta\lambda, \tag{54}$$

which is our second balance relation.

The general equation can be obtained in the same way. Let us summarize our findings in:

**Theorem 8.** *The factorial moment measures of the time stationary SBD satisfy the following balance relations:*

$$\int_{\mathbb{R}^d} \rho_{mi}^{[2]}(\|x\|) f(\|x\|) dx = \lambda, \tag{55}$$

and for all  $k \geq 2$ , for all  $x_1, \dots, x_k$  in  $\mathbb{R}^d$ ,

$$\begin{aligned}
& \rho^{[k]}(x_1, \dots, x_k) \left( \sum_{i=1, k} \sum_{j=1, k, j \neq i} f(\|x_i - x_j\|) \right) \\
& + \int_{\mathbb{R}^d} \rho^{[k+1]}(x_1, \dots, x_k, z) \left( \sum_{i=1, k} f(\|x_i - z\|) \right) dz \\
& = \lambda \sum_{i=1, k} \rho^{[k-1]}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k).
\end{aligned} \tag{56}$$

The general relation ( $k \geq 2$ ) can be re-expressed in terms of the functions  $\rho_{st}^{[k]}$  as

$$\begin{aligned}
& \rho_{st}^{[k]}(x_2 - x_1, \dots, x_k - x_1) \left( \sum_{i=1,k} \sum_{j=1,k, j \neq i} f(\|x_i - x_j\|) \right) \\
& + \int_{\mathbb{R}^d} \rho_{st}^{[k+1]}(x_1 - z, \dots, x_k - z) \left( \sum_{i=1,k} f(\|x_i - z\|) \right) dz \\
& = \lambda \sum_{i=2,k} \rho_{st}^{[k-1]}(x_2 - x_1, \dots, x_{i-1} - x_1, x_{i+1}, \dots, x_k - x_1) \\
& + \lambda \rho_{st}^{[k-1]}(x_3 - x_2, \dots, x_k - x_2). \tag{57}
\end{aligned}$$

## 8.2 Bounds and Approximations

**Repulsion** The next result says that in the stationary regime, there are less points (in terms of their  $f$ -weight) around a typical point (i.e. under the  $\mathbb{P}_\Upsilon^0$ ) than around a typical location of the Euclidean plane (i.e. under the stationary probability  $\mathbb{P}$ ). Note that this  $f$ -repulsion effect differs from what is usually called repulsion (as in e.g. determinantal point processes).

**Theorem 9** ( $f$ -repulsion). *Under the assumptions of Theorem 7, in the stationary regime,*

$$\mathbb{E} \sum_{X \in \Upsilon} f(\|X\|) \geq \mathbb{E}_\Upsilon^0 \sum_{X \in \Upsilon \setminus \{0\}} f(\|X\|). \tag{58}$$

*Proof.* From (29), in steady state

$$\lambda a \beta_\Upsilon = \beta_\Upsilon \mathbb{E}_\Upsilon^0(\pi_\Upsilon^2).$$

From (15), we also have

$$\lambda = \beta_\Upsilon \mathbb{E}_\Upsilon^0(\pi_\Upsilon).$$

Hence

$$a \beta_\Upsilon \mathbb{E}_\Upsilon^0(\pi_\Upsilon) = \mathbb{E}_\Upsilon^0(\pi_\Upsilon^2) \geq \mathbb{E}_\Upsilon^0(\pi_\Upsilon)^2,$$

which directly gives (58).  $\square$

**First Order Approximation** It follows from Theorem 9 that

$$\int_{\mathbb{R}^d} \rho_{st}^{[2]}(u) f(\|u\|) du \leq \beta^2 \int_{\mathbb{R}^d} f(\|u\|) du = \beta^2 a. \quad (59)$$

This and (52) give the bound:

$$\beta \geq \sqrt{\frac{\lambda}{a}}. \quad (60)$$

This can actually be seen as an approximation of order 1 where one (erroneously) pretends that  $\rho_{st}^{[2]}(u) = \beta^2$ . The first order approximation of the intensity is hence

$$\hat{\beta}_1 = \sqrt{\frac{\lambda}{a}} \quad (61)$$

for the intensity and

$$\rho_1^{[k]}(x_1, \dots, x^k) = \hat{\beta}_1^k \quad (62)$$

for the  $k$ -th moment measure.

**Second Order Approximation** Similarly, let us use the following approximation:

$$\begin{aligned} & \int_{\mathbb{R}^d} \rho^{[3]}(x_1, x_2, z) (f(\|x_1 - z\|) + f(\|x_2 - z\|)) dz \\ & \approx \frac{1}{\beta} \int_{\mathbb{R}^d} \rho^{[2]}(x_1, x_2) \rho^{[2]}(x_1, z) f(\|x_1 - z\|) dz \\ & + \frac{1}{\beta} \int_{\mathbb{R}^d} \rho^{[2]}(x_1, x_2) \rho^{[2]}(x_2, z) f(\|x_2 - z\|) dz. \end{aligned}$$

We then get from this and from (53) the (heuristic) equation

$$\begin{aligned} & 2\rho^{[2]}(x_1, x_2) f(\|x_1 - x_2\|) \\ & + \frac{1}{\beta} \rho^{[2]}(x_1, x_2) \int_{\mathbb{R}^d} \left( \rho^{[2]}(x_1, z) f(\|x_1 - z\|) + \rho^{[2]}(x_2, z) f(\|x_2 - z\|) \right) dz \\ & \approx 2\beta\lambda. \end{aligned}$$

This gives:

$$\rho_{mi}^{[2]}(r) \approx \frac{\beta\lambda}{f(r) + \mu}, \quad (63)$$

with

$$\mu = \frac{1}{\beta} \int_{\mathbb{R}^d} \rho_{st}^{[2]}(x) f(\|x\|) dx.$$

Multiplying (63) by  $f(r)$  and integrating leads to the following equation with unknown  $\mu$ :

$$\mu \approx d\nu_d \lambda \int_0^\infty \frac{f(r)}{f(r) + \mu} r^{d-1} dr. \quad (64)$$

The left hand side of the last equation is increasing (in  $\mu$ ) from 0 to infinity whereas the right hand side is strictly decreasing from infinity to 0. Since both functions are continuous, there is one and only one solution to this equation that we will denote by  $\hat{\mu}_2$ . The second order approximation of the reduced second moment density  $g(r) = \frac{1}{\beta} \rho_{mi}^{[2]}(r)$  is then

$$\hat{g}_2(r) = \frac{\lambda}{f(r) + \hat{\mu}_2}, \quad (65)$$

whereas the second order approximation of the density is

$$\hat{\beta}_2 = \lim_{r \rightarrow \infty} \hat{g}_2(r) = \frac{\lambda}{\hat{\mu}_2}. \quad (66)$$

and of course

$$\hat{\rho}_{mi,2}^{[2]}(r) = \frac{\lambda^2}{\hat{\mu}_2(f(r) + \hat{\mu}_2)}. \quad (67)$$

**Third Order Approximation** By the same arguments, the third order approximation is based on the equation

$$\rho^{[3]}(x_1, x_2, x_3) \approx \frac{\lambda (\rho^{[2]}(x_1, x_2) + \rho^{[2]}(x_2, x_3) + \rho^{[2]}(x_1, x_3))}{2(f(\|x_1 - x_2\|) + f(\|x_2 - x_3\|) + f(\|x_3 - x_1\|)) + 3\mu}, \quad (68)$$

with  $\mu$  as defined above. Let

$$\begin{aligned} h(x_1, x_2, z) &= \frac{f(\|x_1 - z\|) + f(\|x_2 - z\|)}{2(f(\|x_1 - x_2\|) + f(\|x_1 - z\|) + f(\|x_2 - z\|)) + 3\mu} \\ j(x_1, x_2) &= \frac{1}{2f(\|x_1 - x_2\|) + \lambda \int_{\mathbb{R}^d} h(x_1, x_2, z) dz}. \end{aligned}$$

Equations (68) and (53) lead to the following Volterra type integral equation for the third order approximation of  $g^{[2]}(x_1, x_2) := \frac{1}{\beta} \rho^{[2]}(x_1, x_2)$ :

$$\begin{aligned} \hat{g}_3^{[2]}(x_1, x_2) &= 2\lambda j(x_1, x_2) \\ &- \lambda j(x_1, x_2) \int_{\mathbb{R}^d} (\hat{g}_3^{[2]}(x_1, z) + \hat{g}_3^{[2]}(x_2, z)) h(x_1, x_2, z) dz. \end{aligned} \quad (69)$$

## 9 Appendix

### 9.1 Proof of Equation (15)

Let  $C$  be a bounded Borel set of Lebesgue measure 1. Let  $t$  be fixed and let  $\epsilon > 0$ . Given  $\Phi'_t$ , denote the points of  $\Phi'_t(C)$  by  $X_1, \dots, X_k$ , and let  $t_1, \dots, t_k$  be independent exponential random variables with parameters  $\pi_{\Phi'_t}(X_1), \dots, \pi_{\Phi'_t}(X_k)$ , respectively. Let  $t_b$  be an independent exponential random variable with parameter  $\lambda$  and let  $N_\epsilon$  be an independent Poisson random variable with the parameter  $\epsilon\lambda$ . We have

$$\begin{aligned}
\mathbb{E}[\Phi'_{t+\epsilon}(C) \mid \Phi'_t] &\leq \Phi'_t(C) + \mathbb{P}[N_\epsilon = 1 \mid \Phi'_t] \mathbb{P}[\min t_i > \epsilon \mid \Phi'_t] \\
&\quad - \sum_{i=1}^k \mathbb{P}[N_\epsilon = 0 \mid \Phi'_t] \mathbb{P}[t_i < \epsilon, \min_{j \neq i} t_j > \epsilon \mid \Phi'_t] \\
&\quad + \mathbb{E}[N_\epsilon \mid \Phi'_t] \mathbb{P}[\text{two or more of } t_b, t_1, \dots, t_k < \epsilon \mid \Phi'_t] \\
&\leq \Phi'_t(C) + e^{-\lambda\epsilon} \lambda\epsilon - \sum_{i=1}^k (1 - e^{-\epsilon\pi_{\Phi'_t}(X_i)}) e^{-\epsilon(\lambda + \sum_{j \neq i} \pi_{\Phi'_t}(X_j))} \\
&\quad + \lambda\epsilon \sum_{i=1}^k (1 - e^{-\lambda\epsilon}) (1 - e^{-\epsilon\pi_{\Phi'_t}(X_i)}) \\
&\quad + \lambda\epsilon \sum_{i \neq j=1}^k (1 - e^{-\epsilon\pi_{\Phi'_t}(X_i)}) (1 - e^{-\epsilon\pi_{\Phi'_t}(X_j)}).
\end{aligned}$$

Hence

$$\begin{aligned}
\mathbb{E}[\Phi'_{t+\epsilon}(C) \mid \Phi'_t] &\leq \Phi'_t(C) + \lambda\epsilon \\
&\quad - \sum_{i=1}^k \epsilon\pi_{\Phi'_t}(X_i) \left(1 - \frac{1}{2}\epsilon\pi_{\Phi'_t}(X_i)\right) \left(1 - \epsilon(\lambda + \sum_{j \neq i} \pi_{\Phi'_t}(X_j))\right) \\
&\quad + o(\epsilon) + \epsilon^2 k \sum_{i=1}^k \pi_{\Phi'_t}(X_i),
\end{aligned}$$

where  $o(\epsilon)$  is deterministic. Taking now expectation w.r.t. the point process  $\Phi'_t$ , we get

$$\begin{aligned}
\beta_{\Phi'_{t+\epsilon}} &\leq \beta_{\Phi'_t} + \lambda\epsilon - \beta_{\Phi'_t} \epsilon \mathbb{E}_{\Phi'_t}^0 \pi_{\Phi'_t}(0) \\
&\quad + \frac{1}{2} \epsilon^2 \mathbb{E} \sum_{X \in \Phi'_t \cap C} \pi_{\Phi'_t}^2(X) + \epsilon^2 \lambda \mathbb{E} \sum_{X \in \Phi'_t} \pi_{\Phi'_t}(X)
\end{aligned}$$

$$\begin{aligned}
& +\epsilon^2 \beta_{\Phi'_t} \mathbb{E} \sum_{X \neq Y \in \Phi'_t \cap C} \pi_{\Phi'_t}(Y) \\
& +o(\epsilon) + \epsilon^2 \mathbb{E} \Phi'_t(C) \sum_{X \in \Phi'_t \cap C} \pi_{\Phi'_t}(X).
\end{aligned}$$

From Lemma 3 (or more precisely from its extension to  $\Phi'$ ),  $\beta_{\Phi'_t} \mathbb{E}_{\Phi'_t}^0 \pi_{\Phi'_t}(0)$  is finite for all  $t$ . More generally, the proof of Lemma 3 can easily be extended to show that each term of the last equation involving an expectation is finite. This implies that

$$\limsup_{\epsilon \rightarrow 0} \frac{\beta_{\Phi'_{t+\epsilon}} - \beta_{\Phi'_t}}{\epsilon} \leq \lambda - \beta_{\Phi'_t} \mathbb{E}_{\Phi'_t}^0 \pi_{\Phi'_t}(0).$$

The inferior limit is derived using similar techniques.

## 9.2 Proof of Equations (32) and (40)

We first recall the general form of the mass transport principle. Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space endowed with a shift  $\theta_u$ ,  $u \in \mathbb{R}^d$ . Let  $N$  and  $N'$  be two  $\theta_u$ -compatible point processes on  $\mathbb{R}^d$ , with respective intensities  $\beta_N$  and  $\beta_{N'}$  and Palm probabilities  $\mathbb{P}_0$  and  $\mathbb{P}'_0$ . Then, for all functions  $g : \mathbb{R}^d \times \Omega \rightarrow \mathbb{R}^+$ , one has

$$\beta_N \mathbb{E}_0 \int_{\mathbb{R}^d} g(y, \omega) N'(dy) = \beta_{N'} \mathbb{E}'_0 \int_{\mathbb{R}^d} g(-x, \theta_x(\omega)) N(dx).$$

We now give the proof for (40); (32) can be obtained exactly the same way.

The R.H.S. in (40) can be rewritten as

$$\beta_{\mathcal{R}_t} |C| \mathbb{E}_{\mathcal{R}_t}^0 \sum_{Y \in \mathcal{Z}_t} f(|Y|) \pi_{\mathcal{R}+\mathcal{Z}+\mathcal{A}_t}(0) = |C| \beta_{\mathcal{R}_t} \mathbb{E}_{\mathcal{R}_t}^0 \int_{\mathbb{R}^d} g(y, \omega) \mathcal{Z}_t(dy),$$

with  $g(y, \omega) = f(|y|) \pi_{\mathcal{R}+\mathcal{Z}+\mathcal{A}_t}(0)$ . From the mass transport principle,

$$\begin{aligned}
\beta_{\mathcal{R}_t} \mathbb{E}_{\mathcal{R}_t}^0 \int_{\mathbb{R}^d} g(y, \omega) \mathcal{Z}_t(dy), &= \beta_{\mathcal{Z}_t} \mathbb{E}_{\mathcal{Z}_t}^0 \int_{\mathbb{R}^d} g(-x, \theta_x \omega) \mathcal{R}_t(dx) \\
&= \beta_{\mathcal{Z}_t} \mathbb{E}_{\mathcal{Z}_t}^0 \sum_{X \in \mathcal{R}_t} f(|X|) \pi_{\mathcal{R}+\mathcal{Z}+\mathcal{A}_t \circ \theta_X}(0) \\
&= \beta_{\mathcal{Z}_t} \mathbb{E}_{\mathcal{Z}_t}^0 \sum_{X \in \mathcal{R}_t} f(|X|) \pi_{\mathcal{R}+\mathcal{Z}+\mathcal{A}_t}(X).
\end{aligned}$$



The L.H.S. in (40) can be rewritten as

$$\begin{aligned}
& \mathbb{E} \sum_{Y \in \mathcal{Z}_t \cap C} \sum_{X \in \mathcal{R}_t} f(|X - Y|) \pi_{\mathcal{R} + \mathcal{Z} + \mathcal{A}_t}(X) \\
&= \mathbb{E} \sum_{Y \in \mathcal{Z}_t \cap C} \sum_{X \in \mathcal{R}_t} f(|X - Y|) \pi_{\mathcal{R} + \mathcal{Z} + \mathcal{A}_t \circ \theta_Y}(X - Y) \\
&= \beta_{\mathcal{Z}_t} |C| \mathbb{E}_{\mathcal{Z}_t}^0 \sum_{X \in \mathcal{R}_t} f(|X|) \pi_{\mathcal{R} + \mathcal{Z} + \mathcal{A}_t}(X),
\end{aligned}$$

which concludes the proof.

### 9.3 Table of Notation

$a = \int_{\mathbb{R}^d} f(\ x\ ) dx$	Strength of the response function
$\mathcal{A}$	Set of antizombies
$\mathcal{A}(z)$	Set of antizombies offspring of $z$
$b_p$	Birth time of point $p$
$\beta_X$	Intensity of the stationary point process $X$ on $\mathbb{R}^d$
$B(x, r)$	Ball of radius $r$ centered in $x$
$d$	Dimension of the Euclidean space
$D$	Convex set of $\mathbb{R}^d$
$d_p$	Death time of point $p$
$\delta_x$	Dirac measure at $x$
$\mathbb{E}_\chi^0$	Palm probability of the $\chi$ point process
$f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$	Response function
$I_{pq}$	Connection direction for $(p, q)$
IS	Investigation stack
$K$	Upper-bound on $f$
$\lambda$	Birth rate
$l^d(C) =  C $	Lebesgue measure on $\mathbb{R}^d$ of the Borel set $C$
$\mu$	Death rate
$\nu_d$	volume of a unit ball
$\Phi_t$	Counting measure of the nodes living at time $t$ for the $\emptyset$ initial condition as obtained by Sheriff
$\Phi'_t$	Counting measure of the nodes living at time $t$ for the $\mathcal{Z}$ initial condition as obtained by Sheriff
$\tilde{\Phi}_t$	Counting measure of the nodes unfinished at time $t$ as obtained by Sheriff $^{\mathcal{Z}}$

$\pi_X(z) = \sum_{x \in X} f( x - z )$	Death pressure exerted on $z$ by the point process $X$ .
$\Pi_X(Y) = \sum_{y \in Y} \pi_X(y)$	Death pressure exerted on the point process $Y$ by the point process $X$
$p, q, \dots$	Points of $\Psi$
$\Psi$	Arrival counting measure on $\mathbb{R} \times \mathbb{R}^d$
$\Psi_t$	Arrival counting measure on $\mathbb{R} \times \mathbb{R}^d$ after time $t$
$\Psi_{(s,t)}$	Arrival counting measure on $\mathbb{R} \times \mathbb{R}^d$ in interval $(s, t)$
$\mathcal{R}$	Set of regular points
RCG	Random Connection Graph
SBD	Spatial Birth and Death
Sheriff	Pathwise construction of the set of nodes living at all $t$
Sheriff <sup>Z</sup>	Simultaneous construction of the set of nodes living at all $t$ , for two different initial conditions
$S_p$	Stack of node $p$
$\mathcal{S}$	Set of special points
$\mathcal{S}(z)$	Set of special points offspring of $z$
$t$	Time
$T_{pq}$	Connection time for $(p, q)$
$x, y, \dots$	Points of $\Phi$
$x_p$	Birth location of point $p$
$\mathcal{Z}$	Set of zombies
$\mathcal{Z}_0$	Initial condition point process
$\mathcal{Z}(z)$	Set of zombies offspring of $z$
$\mathfrak{z}(s)$	Ancestor of the special node $s$

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